A Physicist's Introduction to Topology: Connections to Bosons, Fermions, and Anyons

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The purpose of this paper is as a concise, but intuitive introduction to Topology from the Physicist's perspective, focusing on the Fundamental Group. We will see how the Fundamental Group of a Topological space furnishes a formalism to understand particle statistics in D dimensional space-time.

INTRODUCTION

Loosely, Topology is the study of shapes. This definition may seem redundant, as it seems to be in conflict with the notion of Geometry. However the two fields are actually quite different. The first recorded instances of Geometry have been traced back to Egyptian times as early as 2000 BC. Studies of Geometry entailed measuring things like angles, lengths and various "metric" quantities; things that change when you "stretch" the space.

Mathematicians naturally wanted a more flexible formalism for defining objects, something that was blind to the stretching or deformation of the space. Loosely, they were looking for a kind of mathematics that was invariant under "stretches" and "compressions": the formal analog of an object made of rubber or clay. The kind of identifications we are making here are formally known as *Homeomorphisms*. The objects, or "blobs", are known as Topological Spaces.

My (perhaps ambitious) goal for this paper is to give a brief overview of Topology, with a few examples, and then get to the crux of the paper– explaining how particle statistics can be readily visualized with the help of Topology, and in particular the "Fundamental Group". By the end of the paper, we will have seen how Bosons and Fermions emerge in 4 space-time dimensions, and we will see their analog, *Anyons*, appear when we consider 2 space dimensions.

TOPOLOGY AND THE FUNDAMENTAL GROUP

Basics[1]

Perhaps the best way to understand topology is to consider the historical progression. Mathematicians already knew how to do math on a space with a metric (differential geometry). However, the whole point of Topology was to throw out information about angles and lengths, while preserving information about "closeness". Thus we need to throw out the metric. Formally, a Topological Space is a set S together with a collection of subsets of S, τ. To draw an analogy with the real numbers $\mathbb{R}$ (which themselves can be made into a Topological space), we may call the collection τ the collection of "open sets", and in our analogy can be thought of as the collection of open sets on $\mathbb{R}$, (i.e. unions of open intervals). Notice that when we call these sets "open", we should not imagine that this word comes biased with any of the notions of openness that we already have. We are going to define new axioms that the collection of open sets must have that we will only *model* after our intuition from $\mathbb{R}$. Let's get trivial corner cases out of the way. First off, in $\mathbb{R}$, we had that both the empty set and $\mathbb{R}$ itself were both open. So let's insist in a larger setting that our topological space (S, τ) has the property that $\emptyset, S \in \tau$. Secondly, our intuition from $\mathbb{R}$ tells us that if we take the union of two open sets, we should obtain another open set. So let's insist this for our Topological space as well. We should be able to say the same thing for intersections, with a small caveat. Consider the intersection of open sets on $\mathbb{R}$:

$$\bigcap_{i=1}^{\infty} \left(-\frac{1}{i}, \frac{1}{i} \right) = \{0\} ,$$

or in other words an infinite intersection of open sets centered at zero, with progressively smaller and smaller width. Notice that any non-zero number x, no matter how small, will not be a member of this intersection, since I can always find an interval in the intersection not containing x. 0, however, is always contained, so we conclude that the set consists only of the number 0. This is a problem since $\{0\}$ is not considered open in $\mathbb{R}$. We can circumvent these sorts of issues by only allowing finite intersections. With these three axioms, we have defined the notion of a Topological Space!

Cool Stuff

Now that we understand what a Topological Space is, I'm going to jump way ahead to try to motivate the Fundamental Group. Skipping ahead comes at a price, since we have not developed the formalism necessary to fully define these ideas. But we will be Physicists and use our intuition to grasp the Mathematical ideas. Let's imagine our topological space as some kind of abstract surface, with a notion of closeness defined on it. Imagine dropping a rubber band loop into the space, which formally will be a little loop which passes through the point x_0

in our space. This is actually a function $f : [0,1] \to S$, with the property that $x_0 = f(0) = f(1)$. Now imagine stretching the band around in the Topological Space, keeping the loop passing through x_0. Since we are talking Topology here, any two loops based at the point x_0 will be thought of as *identical* if one can be "stretched" into the other continuously. The notion of "stretchability" is formally called *homotopy* in Mathematical parlance. The wonderful reality is that the collection of all "essentially different" loops in our topological space has a lot of great properties– there is actually a sensible way to add two loops together and obtain a new loop, there's an identity loop and there's even a notion of an inverse loop, and addition of loops is associative. Therefore this collection actually forms a group! Mathematicians call this object the *Fundamental Group*, which is often denoted as π_1.

Let us explore the Fundamental Group with some examples.[2].

1. $\mathbb{R}^n$: Any loop can be contracted down to a point in an n-dimensional plane. Then $\pi_1(\mathbb{R}) = 0$ (trivial group)

2. $\mathbb{S}^0$: The 0-dimensional sphere consists of two disconnected points. Then any loop can be trivially contracted (it already is). Then $\pi_1(\mathbb{S}^0) = 0$. Roughly, the dimensionality is too small to get knotted up.

3. $\mathbb{S}^1$: The 1-dimensional sphere is a circle. Then we can think of many kinds of loops. We may choose to wrap any integer number of times around the circle (even in the reverse direction) Then $\pi_1(\mathbb{S}^1) = \mathbb{Z}$.

4. $\mathbb{S}^n, n > 1$: The n-dimensional sphere for $n > 1$ is difficult to visualize for $n > 2$, but for $n = 2$ the answer should be intuitively clear. Any loop can just be reduced to a single point. It turns out this is always possible in higher dimensions. $\pi_1(\mathbb{S}^n) = 0$ for $n > 1$.

This list may sound rather boring so far, but for the record Mathematicians also have the higher homotopy groups to think about, which encode information about how higher dimensional spheres can be projected down into Topological spaces and then deformed around.

CONNECTION TO PHYSICS [3]

We want to understand how the Physics is elucidated by Topology, particularly in the context of the statistics of particles. Take the simplest example of two identical particles in an n-dimensional space. What is the configuration space $\mathcal{S}$ of this system? Naively, we may expect it to be $\mathbb{R}^n \times \mathbb{R}^n$. However this ignores the indistinguishability. Instead, notice that we may first extract the center-of-mass coordinate as an independent configuration dimension. Therefore, the space decomposes into a cartesian product:

$$\mathcal{S} = \mathcal{E}_n \times r(n,2),$$

where $\mathcal{E}_n$ is the Euclidean space in n dimensions, and $r(n,2)$ is the so-called "relative" space, consisting of the relative particle configurations. Since we don't allow the particles to overlap, we are interested in this relative space which excludes the configuration where the two particles are coincident. Let us examine the nature of this relative space. We can completely characterize the system by defining the position of one of the particles, since then the other particle can be determined as its negative. To this end, we might expect the answer to be $\mathcal{E}_n - \{0\} = \mathbb{S}^{n-1} \times (0,\infty)$. However, since the particles are indistinguishable, we must identify $\mathbf{v} \equiv -\mathbf{v}$. In this identification, we "fold up" the spherical component of the Cartesian product into a projective plane: [3]

$$r(n,2) - \{0\} = \mathbb{R}P^{n-1} \times (0,\infty).$$

We want to describe the theory of Quantum Mechanics without resorting to an unnatural imposition of particle statistics. Instead, we want the statistics to drop out from the Topology. Suppose we want to describe our system with a one-dimensional, complex Hilbert space h_x [2, 3]. This is the space in which our quantum wavefunction $\Psi(\mathbf{x})$ lives. Then when we solve the Schrödinger Equation as usual. However, the configuration space is no longer purely Euclidean, so we need a notion of a Covariant derivative to even write the equation down:

$$D_k \equiv \frac{\partial}{\partial x^k} - ib_k(x),$$

where b_k is a function that is based on the system as well as the choice of gauge. However, the function b_k is not desired at the non-singular points, otherwise we would be introducing a new field, like the vector potential. So we require that b_k be zero away from any singular points. The result of this is ensuring that a parallel transport of our state vector around a point not enclosing a singularity is zero– the path is contractible. On the other hand, if we take a path that encloses a singularity (where the Covariant Derivative is not trivial), Differential Geometry tells us that we observe a non-trivial parallel transport. In this particular instance, the Hilbert space consists of a one-dimensional complex wavefunction, so the change in $\Psi(x)$ around a singularity boils down to a simple phase factor:

$$\Psi(x) \to e^{i\xi}\Psi(x).$$

The interesting thing is that ξ is a free parameter that describes what's happening at the singularity, which encodes information about the particle statistics. If we wrap n times around the singularity, then we pick up n factors. Now here's where Topology kicks in.

Back to Topology Again

In the previous section, we saw how when a system moves around a closed path in configuration space, it sometimes picks up a phase factor. What Topology affords us is the ability to discern when two seemingly different paths acquire the same phase factor. Let's take a more concrete example. We saw that the interesting part of the configuration space of two indistinguishable particles in three dimensions looks like $\mathbb{R}P^2$. First, let's imagine the Fundamental Group for this space. Using the visualization of a 2-sphere with antipodal points identified, let us realize this identification by popping the North and South hemispheres apart, and rotating them relative to each other by a rotation of π. Snap them back together. This is allowed because antipodal points are identified. Now, notice if we deflate, or flatten the sphere so that the North and South poles meet, we automatically identify all the non-equatorial points. We are now left with a disk. We still need to identify the equatorial points, so let's just keep in mind that antipodal points of the disk are to be identified. This is a helpful model of the projective plane. Now let's visualize elements of the Fundamental Group. The trivial element e can be visualized as a simple loop in the middle of the disk somewhere. Can we do anything else? Yes. We haven't used the boundary identification in our loops yet. How about a loop that starts on the boundary, moves across the diameter of the disk to the opposite side, and then pops through the portal right back where it started? A little thinking should convince you that this loop cannot be deformed into the trivial loop– the antipodal points are "pinned", and there is no way to change that.

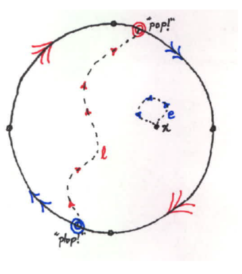

FIG. 1: Two paths from different equivalence classes. e is contractible, where ℓ is not.

One might expect that this group element– call it ℓ– acts as a generator for the Fundamental Group, and so the conclusion is that the Fundamental Group is $\mathbb{Z}$. Let's think a little bit harder just to make sure. What would the element ℓ^2 be? Now our path consists of two traversals of ℓ: two parallel, diametric lines cutting across the disk. As shown in the figure, we are going to consider what happens when we rotate a pair of portals a half-rotation.

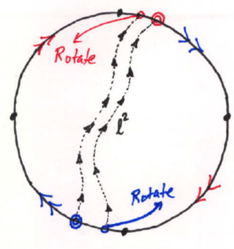

FIG. 2: Paths whose corresponding group element is ℓ^2. This is achieved by traversing the path corresponding to ℓ twice.

If we rotate the portals as shown in FIG. 4, then we get

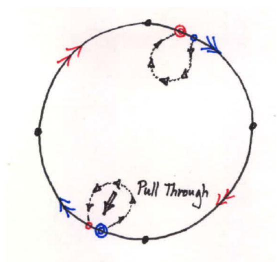

FIG. 3: Trying to untie the path

Then we can just pull the loop through, and get

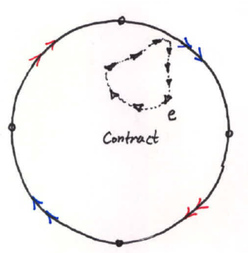

FIG. 4: Paths whose corresponding group element is ℓ^2 are contractible!

With this observation, and some intuition, this tells us something profound– the Fundamental Group of $\mathbb{R}P^2$ is $\mathbb{Z}_{\nvDash}$, consisting of only e and ℓ. This means that whatever factor we pick up when traversing the singularity must disappear upon another traversal. Therefore the factor is either 1 or -1, and these two cases correspond to Bosons and Fermions.

Anyons

It turns out that a similar argument follows in higher dimensions. There are always Bosons and Fermions in n-dimensions for $n \geq 3$. However, something unusual happens when we consider $n = 2$. The peculiar thing that happens is that $\mathbb{R}P^1 = \mathbb{S}^1$. A fantastic way to visualize this is to realize the antipodal identification of the circle with our intuition for rubber bands. Sometimes when you need a tighter band, you twist it into a figure eight, and then fold the two lobes of the eight together. In doing this, you're creating the Real Projective Line because you're bringing antipodal points together. And the band still looks like a band, it just loops around twice now. Now that we've convinced ourselves that $\mathbb{R}P^1$ is S^1, let's consider the Fundamental Group. As discussed in the Topology section, it's easy to visualize why the $\pi_1 S^1 = \mathbb{Z}$. But this means something profound physically– every time we traverse the singularity we really *do* return with a new phase. There's no peculiar identifications that constrain what our phase factor can be. Consequently, rather than falling into two categories like they did in three or more dimensions, types of particles in two dimensions are indexed by a continuous real parameter ξ. Such particles are collectively known as *Anyons*[4].

Acknowledgements

I would like to thank Professor McGreevy for his support and calm composure when explaining concepts to me. I very much appreciate his encouragement, teaching and guidance. I would also like to thank Franciscus Alex Rebro, my Math friend from UC Riverside for many enriching conversations on the rich subject of Differential Geometry and Topology, as well as Group Theory. I would like to thank Shauna Kravec for informing me of the existence of John Preskill's Lecture Notes on the subject of Quantum Computation, which included some Knot Theoretic topics.

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