Entanglement entropy via QFT

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The entanglement entropy is a useful measure of the "degree" of entanglement in a quantum system. Certain systems can effectively be described by a corresponding QFT; in the case of a 1-dimensional system near a phase transition, the entropy can be calculated using a relativistic, massless CFT in 1+1 spacetime dimensions.

INTRODUCTION

Measuring the "amount" of entanglement in a system is a matter of theoretical interest [1], but seems loosely defined on its own. One metric of the entanglement is something called the entanglement entropy; as we'll see, this is a good metric to study, because in certain cases (namely, when the theory is reducible to a *conformal field theory*, or CFT), this is an exactly calculable quantity. Since we could easily be talking about systems which aren't in the realm of QFT, for example a chain of spins, you might ask why I'm bothering to mention field theory at all. It turns out to be useful for such systems near critical points in their phase diagrams, in which case correlation lengths grow much larger than any microscopic "graininess" like a lattice spacing.

I'll start by defining the entanglement entropy, then talk about CFT a little bit before moving on to a specific example due to [4], in which case it is clear how the machinery of CFT is applicable and powerful in the computation of the entropy.

ENTANGLEMENT ENTROPY

Imagine subdividing a quantum system into several parts [1]; for now, let's say two, called A and B. Also imagine that the system $A \cup B$ is a pure quantum state $|\Psi\rangle$, so that we can define the density matrix $\rho = |\Psi\rangle\langle\Psi|$. We can define the reduced density matrix for either subsystem by tracing out the degrees of freedom of its complement à la

$$\rho_A = \text{Tr}\,_B \rho \tag{1}$$

and vice versa for ρ_B. The entanglement entropy is defined for subsystem A, for example, in an identical manner to the von Neumann entropy as $S_A = -\text{Tr}\,\rho_A \log \rho_A$. To (heavily) paraphrase Holzhey, et. al. [4], this entropy describes correlations between A and $A \cup B$ by counting the number of states in B consistent with measurements restricted to A, given a priori knowledge that $A \cup B$ is a pure state.

Let's think about a specific system [1]; namely, a set of local commuting observables $\{\hat{\phi}(x_i)\}$ living on a one-dimensional lattice with sites labeled by the x_i, with lattice spacing a. The domain of x has length L, which may be finite or (semi-)infinite. At finite inverse temperature β, the density matrix is defined by

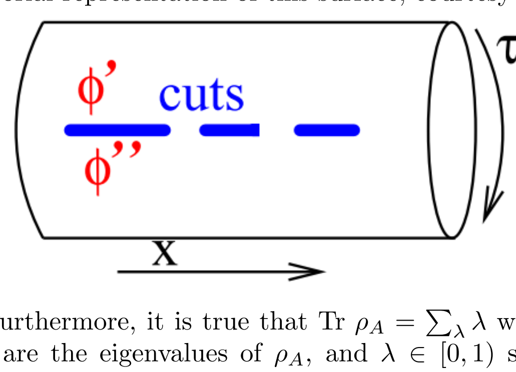
$$\rho_{ij} = \frac{1}{Z(\beta)} \langle \{\phi_i(x_i)\}| e^{-\beta \hat{H}} |\{\phi_j(x_j)\}\rangle. \tag{2}$$

$Z(\beta)$ is the partition function, defined in the usual way as $Z(\beta) \equiv \text{Tr}\; e^{-\beta \hat{H}}$. Here, the trace becomes a path integral over the collection of ϕ's; explicitly

$$\rho = \frac{1}{Z} \int [d\phi(x,\tau)] \prod_x \delta(\phi(x,0) - \phi_i(x_i)) \times \prod_x \delta(\phi(x,\beta) - \phi_j(x_j)) e^{-\int_0^\beta d\tau L_E}. \tag{3}$$

If we want $Z(\beta)$, it's just the numerator of (2) but with the identification $\phi_i(x) = \phi_j(x)$; in the path integral this has the effect of "stitching" together the integration surface at $\tau = 0$ and $\tau = \beta$, so it now looks like a tube with circumference β.

At some point we want to take a subset of our domain, which might be a collection of intervals $(u_1, v_1) \ldots (u_N, v_N)$; call it A. To calculate the reduced density matrix ρ_A, we should only stitch the edges of the integration surface in (3) at values of x which are *not* contained in A. This leaves open cuts on our tube. If we join together n copies of this tube (bear with me, there is a goal in mind) at the cuts, i.e. in such a way that we identify $\phi_i^k(x) = \phi_j^{k+1}(x)$ for all $x \in A$, where $1 \le k \le n$ i.e. k labels the tube, this is like taking the trace of n powers of ρ, i.e. Tr ρ_A^n. The figure below is a really nice pictorial representation of this surface, courtesy of [2].



Furthermore, it is true that Tr $\rho_A = \sum_\lambda \lambda$ where the λ's are the eigenvalues of ρ_A, and $\lambda \in [0, 1)$ since it's a density matrix; this means Tr $\rho_A^n = \sum_\lambda \lambda^n$ converges and is thus an analytic function for $n > 1$. In addition, using the usual rule for derivatives of an exponential, we find that

$$S_A = -\mathrm{Tr}\ \rho_A \log \rho_A = -\lim_{n\to 1^+} \frac{\partial}{\partial n} \mathrm{Tr}\ \rho_A^n \tag{4}$$

which is well-defined because of the analyticity of Tr ρ_A^n. Finally, if we define the partition function on the n-sheet (comprised of the n stitched-up tubes) to be $Z_n(A)$, then we arrive at the result

$$\mathrm{Tr}\ \rho_A^n = \frac{Z_n(A)}{Z^n} \tag{5}$$

$$\Rightarrow S_A = -\lim_{n\to 1^+} \frac{\partial}{\partial n} \frac{Z_n(A)}{Z^n} \tag{6}$$

where the first equality holds by definition. This is a nice result, because it lets us calculate S_A without ever having to calculate a density matrix (which, in general, may be very difficult or impossible to compute).

We can make this look like quantum field theory by taking the limit $a \to 0$. As we'll see, this leads to terms in S which are proportional to $\log a$. This looks like a problem, but in fact this dependence on a can be shown to drop out of the final expression. In the section after next, we'll focus on a relativistic and massless 1+1 dimension field theory, which is writable as a so-called conformal field theory, or CFT.

WHAT IS A CFT?

Before moving on, it's worth noting what a CFT actually *is*. A CFT exhibits conformal symmetry in the sense that the action of the theory is invariant under conformal transformations. The conformal group includes the translation, rotation, and dilatation (scaling) transformations (remember that a Lorentz transformation is like a rotation after we analytically continue to Euclidean time!). As you may have learned long ago, these are the transformations which preserve angle measures between intersecting lines.

As an example [3], consider the two-point function for a scalar field, $\langle \phi(x)\phi(y)\rangle$. If the theory exhibits conformal symmetry, then under a scaling of the coordinates $x \to \lambda x$, the two-point function should change at most up to a scale factor:

$$\langle \phi(x)\phi(y)\rangle = \lambda^{2\Delta} \langle \phi(\lambda x)\phi(\lambda y)\rangle \tag{7}$$

where Δ is the *scaling dimension* for ϕ. In fact, conformal symmetry imposes translational and rotational ivariance as well, so that two-point functions must be of the form

$$\langle \phi(x)\phi(y)\rangle = |x-y|^{-2\Delta} \tag{8}$$

(up to the normalization of ϕ).

Consider [5] a coordinate transformation of the form $x^\mu \to x^\mu + \varepsilon^\mu(x)$. It turns out that if this is a conformal transformation, $\varepsilon^\mu(x)$ is restricted by the following conditions:

$$\left[g_{\mu\nu}\Box + (d-2)\partial_\mu\partial_\nu\right] \partial \cdot \varepsilon = 0 \tag{9}$$

where d is the number of dimensions (these conditions holds specifically if we take the flat-space case $g_{\mu\nu} = \delta_{\mu\nu}$). Is $d = 2$ somehow special? Answer: yes, it is! Eqs. (9) reveal themselves to be the Cauchy-Riemann equations. The conformal transformations in $d = 2$, therefore, can be written as holomorphic functions on z and $\bar{z}$, under the substitutions $z = x^1 + ix^2$ and $\bar{z} = x^1 - ix^2$. Soon, we'll use z and $\bar{z}$ with the Euclidean space-time coordinates; it turns out this is a useful construct.

AN EXAMPLE: RELATIVISTIC, MASSLESS 1+1-DIM THEORY

This example is originally due to [4], however this discussion follows [1] which is much clearer and is largely a "redo" of the original treatment. More specifically than the section title would imply, this is a case in which the system in question is restricted to a single interval on the lattice with length ℓ. Before we jump into this specific example, however, we should discuss the nature of the stress tensor in CFT.

After analytic continuation [5] from Euclidean spacetime to $z, \bar{z}$, we can write the stress tensor as a function of these complex variables. Additionally, we can expand the insertion of T at two points z and w, the form of which is general and what is term-by-term allowed by conformal symmetry:

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)}, \tag{10}$$

which is the *operator product expansion* (OPE) for $T(z)T(w)$. The first term is called the *conformal anomaly*, characterized by the central charge c. This term is anomalous because, classically, the expectation value of T should vanish. In fact, $c = 0$ classicaly, so the first term is purely a quantum effect.

The central charge is actually a pretty remarkable thing. One way to think about [3] it is that it's related to a "soft" conformal symmetry breaking which arises from the introduction of a macroscopic scale into the system, which could be a boundary condition, for example. Even more interestingly, c lumps theories together into large *universality classes*; for example, $c = 1$ for the free boson, $c = 1/2$ for the free fermion, etc. In some sense, it's an extensive measure of the degrees of freedom in the system. We'll see how this manifests in the calculation of the entanglement entropy.

Eq. (10) can be written in terms of the generator of the transformation $w \to z$, then integrated to give the transformation of T in the form

$$T(z) \to T(w) = \left(\frac{dz}{dw}\right)^2 T(z) + \frac{c}{12} S[z;w] \tag{11}$$

where $S[z;w]$ is the Schwartzian derivative,

$$S[z;w] = \frac{(\partial_w z)(\partial_w^3 z) - \frac{3}{2}(\partial_w^2 z)^2}{(\partial_w z)^2} \tag{12}$$

where you should remember that $z = z(w)$.

We want to know about the stress tensor in the first place because (5) is just the vacuum expectation value $\langle 0|0\rangle_{R_n}$ on the stitched-up Riemann n-surface; but this is just $\langle T(w)\rangle_{R_n}$. We can do even better by finding a transformation that sends us to some z in which we know the expectation value of $T(z)$ to be zero, eliminating it from the expectation value of (11)! Namely, we can send the Riemann n-sheet to the plane $\mathbb{C}$, in which case we know $\langle T(z)\rangle_{\mathbb{C}} = 0$ by rotational invariance. The mapping that does the trick [1] is

$$w \to z = \left(\frac{w-u}{w-v}\right)^{1/n} \tag{13}$$

where u and v are the branch points on the Riemann surface at the edges of the interval in consideration (under z they get sent to 0 and ∞, respectively).

Using the *conformal* Ward identities, one can also calculate a correlator like $\langle T(w)\phi_n(u)\phi_{-n}(v)\rangle_{\mathbb{C}}$, where $\phi_{\pm n}$ are two operators with the same (complex) scaling dimension $\Delta_n = \bar{\Delta}_n$, and are normalized so that $\langle \phi_n(u)\phi_{-n}(v)\rangle_{\mathbb{C}} = |v-u|^{-4\Delta_n}$, with

$$\Delta_n = \frac{c}{24}\left[1 - \left(\frac{1}{n}\right)^2\right]. \tag{14}$$

The details of the intervening calculation is not as important as the result, which is that

$$\frac{Z_n(A)}{Z^n} = \langle T(w)\rangle_{R_n} = \frac{\langle T(w)\phi_n(u)\phi_{-n}(v)\rangle_{\mathbb{C}}}{\langle \phi_n(u)\phi_{-n}(v)\rangle_{\mathbb{C}}}. \tag{15}$$

This is good, because as it turns out, the RHS is actually an expression that is totally in terms of w, the branch points u, v and the scaling dimension Δ_n! This gives us the LHS of (15) up to a constant c_n:

$$\text{Tr } \rho_A^n = c_n \left(\frac{v-u}{a}\right)^{-(c/6)(n-1/n)} \tag{16}$$

where a has been inserted for dimensional reasons. The c_n's aren't determined, but for $n=1$ this constant should be 1 (since ρ_A is normalized). The result, after using the derivative trick, is that

$$S_A = \frac{c}{3}\log\frac{\ell}{a} + c_1' \tag{17}$$

where c_1' is some constant which is not universal (unlike the central charge!). Notice that a length scale has been introduced into the expression. As we take $a \to 0$, we might think of this as taking the lattice spacing to zero. The expression diverges logartihmically; in a crude sense, it's like we're trying to add up an infinite number of degrees of freedom. This issue can be eliminated in some cases with the proper choice of renormalization factor [4]. In fact, this amounts to redefining the entropy with respect to the ground state, since differences in entropies are much more well-behaved. In this case, even c_1' will disappear.

We can redo the previous calculation, but at a finite inverse temperature β, and obtain instead the result

$$S_A(\beta) \sim \frac{c}{3}\log\left[\frac{\beta}{\pi a}\sinh\frac{\pi\ell}{\beta}\right] + c_1' \tag{18}$$

in which case we recover the previous result in the low-temperature limit $\ell \ll \beta$. In the high-temperature limit $\ell \gg \beta$, the entropy goes to $\sim (\pi c/3)(\ell/\beta)$, which is an extensive quantity, comfortingly in agreement with our classical expectations! We have thus recovered a result which, in some sense, is a connection between the classical and quantum regimes.

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[1] P. Calabrese and J. L. Cardy, J. Stat. Mech. **0406**, P06002 (2004) [hep-th/0405152].

[2] P. Calabrese and J. L. Cardy, Int. J. Quant. Inf. **4**, 429 (2006) [quant-ph/0505193].

[3] P. Di Francesco, P. Mathieu and D. Senechal, New York, USA: Springer (1997) 890 p

[4] C. Holzhey, F. Larsen and F. Wilczek, Nucl. Phys. B **424**, 443 (1994) [hep-th/9403108].

[5] S. V. Ketov, Singapore, Singapore: World Scientific (1995) 486 p