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Pair Creation in an Electric Field

An interesting result of vacuum polarization is the possibility of pair creation of charged particles at sufficiently high field strength. This can be viewed as a tunneling process in which the particles tunnel through a barrier of depth $2m$ to a region where the classical potential due to the gauge field is $-2m$. The results for QED were derived by Schwinger in 1950.

Consider a Dirac fermion field ψ coupled to a spin 1 gauge field A_μ in 3 + 1 dimensions as we have in quantum electrodynamics. Consider the vacuum to vacuum process in the presence of an external field. The amplitude for this process is

$$Z[A]=e^{iW}=\int [d\psi][d\overline{\psi}]\exp[\int d^4x\overline{\psi}[\gamma\cdot(P-eA)-m+i\epsilon]\psi]$$

The probability for this process, that is for no pair creation, is

$$|Z[A]|^2=|e^{iW}|^2=e^{-2\,\mathrm{Im}\,W}$$

So the probability that a pair of fermions will be created is approximately twice the imaginary part of the effective action.

To get an expression for W, we write Z as a functional determinant and replace the determinant by the exponential of a trace of a log. Subtracting off the zero external field contribution, we have

$$W[A]=i\,\mathrm{Tr}\log\left[\frac{\gamma\cdot(P-eA)-m+i\epsilon}{\gamma\cdot P-m+i\epsilon}\right]$$

By taking the transpose inside the trace and applying the identity $\gamma_\mu^T=-C\gamma_\mu C^{-1}$, where C is charge conjugation, we can rewrite the effective action as

$$W[A]=i\,\mathrm{Tr}\log\left[\frac{\gamma\cdot(P-eA)+m-i\epsilon}{\gamma\cdot P+m-i\epsilon}\right]$$

Summing the two expressions, we get, $W[A]=\frac{i}{2}\mathrm{Tr}\log\left(\frac{[\gamma\cdot(P-eA)]^2-m^2+i\epsilon}{P^2-m^2+i\epsilon}\right)$.

Now we use an integral representation of the log in order to evaluate the trace:

$$\log\frac{a}{b}=\int_0^\infty\frac{ds}{s}\left(e^{is(b+i\epsilon)}-e^{is(a+i\epsilon)}\right)$$

Removing the spatial integrals involved in taking the trace and taking twice the imaginary part, we arrive at the probability density for pair creation, $w(x)$. While before Tr represented a trace over spacial states and spinor indices, tr will represent the trace over only spinor indices.

$$w(x)=\mathrm{Re}\,\mathrm{tr}\int_0^\infty\frac{ds}{s}e^{-is(m^2-i\epsilon)}\langle x|\exp\left(is\left[(P-eA(x))^2+\frac{e}{2}\sigma_{\mu\nu}F^{\mu\nu}\right]\right)-e^{isP^2}|x\rangle$$

Where $\sigma_{\mu\nu}=\frac{1}{2}i[\gamma_\mu,\gamma_\nu]$. From here, the specific form for the background field A is used to complete the calculation. For constant electric or magnetic field, we may boost into the frame where the field is either purely magnetic or purely electric. In the magnetic case, W is purely real and no pair creation process may occur. In the electric case, chose a frame where in one spatial component $A = Et$

and in all others $A = 0$. By use of a complete set of momentum states on either side of the operator in the matrix element, we may reduce the problem to that of calculating the trace of the time evolution operator of a simple harmonic oscillator with imaginary frequency $2ieE$. The integral over s may be completed using a Wick rotation to the imaginary axis. We end up with the nonperturbative result

$$w = \frac{\alpha E^2}{\pi^2} \sum_{n=1}^{\infty} n^{-2} \exp\left(\frac{-n \pi m^2}{|eE|} \right)$$

Each term in the sum is suppressed by higher powers of $\exp(-m^2/|eE|^2)$. This is related to the result we expect from treating this as a tunneling process. The system starts at the energy $-2m$ and tunnels through a distance $2m/|eE|$ to reach another classical turning point. The WKB amplitude for this process is

$$\exp(-2 \int_0^{2m/|eE|} dx [2m(2m - |eE|x)]^{\frac{1}{2}}) = \exp(-\frac{8}{3} \frac{m^2}{|eE|})$$

This gets the essential factor right, but only predicts the first term in the series found for the probability. The ratio $m^2/|eE|$ can also be thought of as the creation energy $2m$ over the work done by the field moving the charge over a Compton wavelength, $|eE| \cdot 1/m$. This may be thought of as the work done to bring the particles on shell.

Pair creation is difficult to observe directly because of the strength of electric field required. To reach a ratio $m^2/|eE|$ of order 1 for electrons, the electric field must be 10^7 times larger than that found in a hydrogen atom at the typical location of an electron.

The essential part of the result, that the probability density is suppressed by a sum of exponentials of $-m^2/|eE|$, is independent of the fields being fermionic and of the dimensionality of the field theory. For scalar bosons in 3 + 1 dimensions, the probability is

$$w = \frac{\alpha E^2}{2\pi^2} \sum_{n=1}^{\infty} (-1)^{n+1} n^{-2} \exp\left(\frac{-n \pi m^2}{|eE|} \right)$$

For fermions in 2 + 1 dimensions, the probability becomes

$$w = \frac{(eE)^{\frac{3}{2}}}{4\pi^2} \sum_{n=1}^{\infty} n^{-\frac{3}{2}} \exp\left(\frac{-n \pi m^2}{|eE|} \right)$$

This is unsurprising because the tunneling argument remains unchanged, and the particles are subjected to the same classical potential in the electric field.

References

C. Itzykson and J. Zuber, *Quantum Field Theory* (1980), 193

J. Schwinger, *On Gauge Invariance and Vacuum Polarization* (1950)

H. Hamber, *Quantum Gravitation, The Feynman Path Integral Approach* (2009), 152

G. Dunne, *Planar QED in Magnetic or Electric Solitonic Backgrounds* in *Solitons, Properties, Dynamics, Interactions, Applications* (1997), 27