8.821 F2008 Problem Set 4 Solutions

Nabil Iqbal

I. ONE MORE SYMMETRY

In this problem we will demystify this peculiar "inversion" that is used to such powerful effect by Witten. Consider AdS_{d+1} in Poincare coordinates:

$$ds^2 = \frac{dz^2 + d\vec{x}^2}{z^2} = \frac{dx^A dx_A}{z^2} \tag{1}$$

where $x^A = (z, x^\mu)$ and $x^A x_A \equiv z^2 + x^\mu x_\mu$. Now I claim that the horribly violent inversion

$$x^A \to \bar{x}^A = \frac{x^A}{x^B x_B} \tag{2}$$

is actually an isometry of AdS, meaning that the metric in the barred coordinates takes precisely the same form as it did in the unbarred coordinates. This is not hard to verify, starting with the fact that

$$d\bar{x}^A = \frac{1}{x^B x_B}\left(dx^A - 2\frac{x^A}{x^B x_B} x_C dx^C\right) \tag{3}$$

It is now easy to show that

$$ds^2 = \frac{d\bar{x}^A d\bar{x}_A}{\bar{z}^2} = \frac{(x^D x_D)^2}{z^2}\frac{1}{(x^B x_B)^2}\left(dx^A - 2\frac{x^A}{x^B x_B}x_C dx^C\right)\left(dx_A - 2\frac{x_A}{x^B x_B}x_C dx^C\right) \tag{4}$$

Expanding out the bracket we obtain

$$ds^2 = \frac{d\bar{x}^A d\bar{x}_A}{\bar{z}^2} = \frac{1}{z^2}\left(dx^A dx_A - 4\frac{(x_A dx^A)^2}{x^B x_B} + 4\frac{x_A x^A}{(x^B x_B)^2}(x_C dx^C)^2\right) = \frac{dx^A dx_A}{z^2} \tag{5}$$

as claimed.

II. SCHRODINGER DESCRIPTION OF ADS INSTABILITIES

A. Wave equation in Schrodinger form

In this problem we will derive the celebrated Breitenlohner-Freedman bound, which tells us that a negative mass for a bulk scalar field in AdS_{d+1} does not necessarily lead to any instabilities, provided that it isn't *too* negative. We will work in units where $L_{AdS} = 1$ and the metric is given by (1) but with Lorentzian signature on the constant-z slices. The bulk scalar action is

$$S = -\frac{1}{2}\int d^{d+1}x\sqrt{-g}\left((\nabla\phi)^2 + m^2\phi^2\right) = -\frac{1}{2}\int d^{d+1}x\, z^{1-d}\left(\eta^{AB}\partial_A\phi\partial_B\phi + \frac{m^2}{z^2}\right) \tag{6}$$

The equation of motion can be easily obtained from here to be

$$\phi'' + \frac{(1-d)}{z}\phi' - \left(k^2 + \frac{m^2}{z^2}\right)\phi = 0 \tag{7}$$

where I have assumed a field theory spacetime dependence e^{ikx}, so k^2 is the momentum $-\omega^2 + \vec{k}^2$. We would like to write this equation in a form that makes it look like a Schrodinger equation from nonrelativistic quantum mechanics, i.e. we want to make the pesky term multipling ϕ' go away. To do this, we write $\phi(z) = B(z)\psi(z)$ and plug in; after the dust settles we get

$$B\psi'' + \left(\frac{(1-d)B + 2zB'}{z}\right)\psi' + \frac{1}{z^2}\left[-(m^2 + k^2z^2)B + z(d-1)B' + z^2B''\right]\psi = 0 \tag{8}$$

The term multiplying ψ' can be made to vanish if

$$\frac{B'}{B} = \frac{d-1}{2z} \to B = z^{\frac{d-1}{2}} \times \text{const} \tag{9}$$

We take the arbitrary constant to be 1 and plug in this form for B into the equation of motion. After the derivatives in the last term have exhausted themselves we obtain finally

$$-\psi'' + \left[\vec{k}^2 + \frac{1}{z^2}\left(m^2 - \frac{1-d^2}{4}\right)\right]\psi = \omega^2\psi \tag{10}$$

This is a perfectly normal looking Schrodinger equation, with potential $V(z)$ given by the quantity in square brackets and "energy" given by ω^2. Note that a negative-"energy" solution corresponds to imaginary ω, which means that the solution is growing exponentially in time; this is the instability we are looking for. For the remainder of this problem we will set $\vec{k} = 0$.

Now this differential equation will always have solutions for all values of ω, both real and imaginary, but not all of them will be "normalizable". A normalizable solution to the AdS wave equation is one that has finite energy, in a sense that we will now make precise; in particular, normalizable in the bulk AdS wave equation sense will be closely related to normalizability in the usual QM sense (i.e. $\int dz|\psi(z)|^2 < \infty$).

B. Normalizable solutions to the AdS wave equation

We begin by first noting that if any spacetime has an isometry generated by a Killing vector ξ^μ, the current $j^\nu = T^{\mu\nu}\xi_\mu$ (where $T_{\mu\nu}$ is the stress tensor is covariantly conserved: $\nabla_\nu j^\nu = 0$, as can be easily checked using the Killing equation and the conservation of $T_{\mu\nu}$. Thus given a region R of $d+1$ dimensional space

$$0 = \int_R d^{d+1}x\sqrt{-g}\nabla_\nu j^\nu = \int_{\partial R} d^d x\sqrt{h}n^\mu\xi^\nu T_{\mu\nu} \tag{11}$$

where ∂R is the boundary of R and h is the determinant of the induced metric on this boundary. Let us now specialize to AdS_{d+1} and let the Killing vector be $\xi = \partial_t$. Let us also take R to be a giant chunk of AdS, extending across all space $z \in [0,\infty)$ but bounded in the past and the future by two spacelike slices at t_i and t_f, i.e. $t \in [t_i, t_f]$. In that case ∂R has three components; the two spacelike slices at $t = t_i, t_f$, and the timelike slice at the AdS conformal boundary at $z = 0$ (I will assume that all fields decay away exponentially at $z \to \infty$; the boundary term at $z = 0$ will turn out to be important). This integral then becomes

$$\int_0^\infty dz\ z^{1-d}T_{tt}\Big|_{t=t_i}^{t=t_f} - \int_{t_i}^{t_f} dt\ z^{1-d}T_{tz}\Big|_{z=0} = 0 \tag{12}$$

The second term is the flux of energy-momentum out the AdS boundary; if this is zero then this equation implies that the first integral has the same value at t_f and at t_i and should be interpreted as the statement of energy conservation. Now let us explicitly work out what the relevant components of $T_{\mu\nu}$ are in our case. For a scalar field with action (6) the stress-energy tensor is

$$T_{\mu\nu} = \nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\left[g^{\rho\sigma}\nabla_\rho\phi\nabla_\sigma\phi - m^2\phi^2\right] \tag{13}$$

I would now like to write this in terms of ψ. This is mostly very easy except for the term in $(\partial_z\phi)^2$, which we manipulate with an integration by parts as follows

$$\int_0^\infty dz z^{1-d}(\partial_z\phi)^2 = \phi z^{1-d}\partial_z\phi\Big|_{z=0}^{z=\infty} - \int_0^\infty dz\ \phi\partial_z(z^{1-d}\partial_z\phi) = -\phi z^{1-d}\partial_z\phi\Big|_{z=0} - \int_0^\infty dz\ \psi\left[\partial_z^2\psi + \frac{1-d^2}{4z^2}\psi\right] \tag{14}$$

Similarly, we will manipulate the second term in (12) with a similar integration by parts (except on t)

$$-\int_{t_i}^{t_f} dt\ z^{1-d}T_{tz}\Big|_{z=0} = -\frac{1}{2}\left[\int_{t_i}^{t_f} dt\ z^{1-d}(\partial_t\phi\partial_z\phi - \phi\partial_t\partial_z\phi) + z^{1-d}\phi\partial_z\phi\Big|_{t=t_i}^{t=t_f}\right]_{z=0} \tag{15}$$

Putting the pieces together to assemble (12), we see that the boundary terms from each of the two parts cancel and we obtain the following equation

$$\int_0^\infty dz \frac{1}{2}\left[\partial_t\psi\partial_t\psi + \psi\left(-\partial_z^2 + V(z)\right)\psi\right]\Big|_{t=t_i}^{t=t_f} - \frac{1}{2}\left[\int_{t_i}^{t_f} dt\; z^{1-d}(\partial_t\phi\partial_z\phi - \phi\partial_t\partial_z\phi)\right]_{z=0} = 0 \tag{16}$$

where $V(z)$ is the potential from the Schrodinger equation (10)

$$V(z) = \frac{1}{z^2}\left(m^2 - \frac{1-d^2}{4}\right) \tag{17}$$

Now let us take a step back and think about what we require of a solution to the wave equation. It seems reasonable to require that the energy (defined by the first integral above) is finite, and also that it is *conserved.* We have seen that energy conservation requires that the second term vanish above vanish. Introducing a Fourier expansion in ω, $\phi(t) = \int d\omega\phi(\omega)e^{i\omega t}$, we see that this implies that

$$\int d\omega'\; d\omega\; \int_{t_i}^{t_f} dt e^{it(\omega+\omega')}\left[\omega\phi(\omega)\partial_z\phi(\omega') - \omega'\phi(\omega)\partial_z\phi(\omega')z^{1-d}\right]_{z=0} = 0 \tag{18}$$

Note that the fact that this must hold for all t_i, t_f essentially means it must hold for all ω, ω'. To put this into a form that will turn out to be more useful, we swap ω and ω' in the second term, to obtain

$$\int d\omega'\; d\omega\; \omega z^{1-d}\left[\phi(\omega)\partial_z\phi(\omega') - \phi(\omega')\partial_z\phi(\omega)\right]_{z=0} = 0 \tag{19}$$

For our final trick, we write ϕ in terms of ψ and notice that the reality of $\phi(t)$ implies that $\phi(\omega) = \phi(-\omega)^*$ to write this equation as

$$\psi(\omega)^*\partial_z\psi(\omega') - \psi(\omega')\partial_z(\psi(\omega)^*) = 0 \quad \text{at} \quad z = 0 \tag{20}$$

This equation embodies the statement that "nothing important is leaving through the AdS boundary," and will turn out have rather spectacular consequences in the quantum mechanics problem that we will solve next.

Let us return now to the criterion of "finiteness of the energy". This simply requires that the first bracketed term in (16) is finite. Evaluating it at arbitrary t, we see that the energy is simply

$$E(t) = \frac{1}{2}\int d\omega' d\omega dz\left[-\omega\omega'\psi(\omega)\psi(\omega') + \psi(\omega)\left(-\partial_z^2 + V(z)\right)\psi(\omega')\right]e^{it(\omega+\omega')} \tag{21}$$

We have so far nowhere used the fact that we are evaluating this energy functional *on-shell*, that is, on a solution to the equations of motion (10). Using this fact, noticing that it is exactly the Schrodinger operator that appears in the second term above, and switching $\omega \to -\omega$, we end up with the following expression for the energy

$$E(t) = \frac{1}{2}\int d\omega' d\omega dz\left[+\omega\omega' + \omega'^2\right]\psi(\omega)^*\psi(\omega')e^{it(\omega'-\omega)} \tag{22}$$

This is promising. Now let us think very hard; the Schrodinger operator in (10) is Hermitian (Right? Right. Ignore that prickling feeling at the back of your neck...) Thus its eigenfunctions with different eigenvalues are orthogonal. These eigenfunctions are nothing but the $\psi(\omega)$, and so we should have $\int dz\psi(\omega)^*\psi(\omega') = \delta(\omega-\omega')$. Thus we find

$$E(t) = \int d\omega dz \omega^2|\psi(\omega)|^2 \tag{23}$$

And so finiteness of the energy requires that the wavefunctions $\psi(\omega)$ have finite norm in the usual quantum mechanical sense.

Wait. Something is wrong; I just convinced you that the energy will be conserved only if some awkward boundary condition (20) was met, yet I just found an explicit formula for the energy that is *time-independent*! Why would I lie to you in this way? Well, I assumed that the Schrodinger operator appearing in (10) is Hermitian–is this necessarily true? Consider the operator acting on two eigenfunctions with real eigenvalues

$$\left[-\partial_z^2 + V(z)\right]\psi(\omega) = \omega^2\psi(\omega) \qquad \left[-\partial_z^2 + V(z)\right]\psi^*(\omega') = \omega'^2\psi^*(\omega') \tag{24}$$

Now multiply the first equation by $\psi^*(\omega')$, the second by $\psi(\omega)$, subtract the second from the first, and integrate them both over z. We obtain

$$-\int_0^\infty dz\left[\psi^*(\omega')\partial_z^2\psi(\omega)-\psi(\omega)\partial_z^2\psi^*(\omega')\right]=\left(\omega^2-\omega'^2\right)\int_0^\infty dz\;\psi^*(\omega')\psi(\omega) \tag{25}$$

If $\omega \neq \omega'$, the eigenfunctions have different eigenvalues and the expression on the right-hand side is nonzero and proportional to the inner product of the eigenfunctions. If this operator is Hermitian, these eigenfunctions must be orthogonal, which means that the left-hand side of this expression better vanish. However, if we integrate by parts on this left-hand side, we see that it is not zero but in fact equal to a boundary term

$$-\left[\psi(\omega')^*\partial_z\psi(\omega)-\psi(\omega)\partial_z(\psi(\omega')^*)\right]\Big|_{z=0}=\left(\omega^2-\omega'^2\right)\int_0^\infty dz\;\psi^*(\omega')\psi(\omega) \tag{26}$$

Thus the Schrodinger operator is not Hermitian unless this particular boundary term is zero. This boundary term is precisely the one that we found earlier in (20), and is proportional to the energy flux out the boundary of AdS. Suddenly it all makes sense, and we summarize our findings below:

1. If the Schrodinger operator corresponding to the AdS wave equation is Hermitian on a particular set of eigenfunctions, then those eigenfunctions correspond to classical field configurations with a bulk AdS energy (i.e. the charge under the bulk time translational Killing vector) that is conserved. The condition for Hermiticity is precisely equivalent to the condition that no energy leak out the AdS boundary.

2. Furthermore, if a particular eigenfunction is normalizable in the normal QM sense (i.e. $\int dz|\psi(z)|^2<\infty$), then it corresponds to a classical field configuration that has finite bulk AdS energy.

We now finally have a well-posed quantum mechanics problem; find normalizable negative-energy eigenstates of the Schrodinger operator (10) that satisfy the boundary condition (20).

C. Good old-fashioned Quantum Mechanics

The Schrodinger problem we are solving is

$$\left(-\partial_z^2+\frac{\alpha}{z^2}\right)\psi=-\beta^2\psi \tag{27}$$

where $\alpha=\left(m^2-\frac{1-d^2}{4}\right)$. We now want to examine the behavior of the solutions as a function of α; in particular, we are seeking negative-energy solutions, which with the convention used in (27) corresponds to real β. I should point out that this particular potential has a venerable history, ranging from a paper from 1950 [1] (which I am largely following) to the recent work of our own B. Swingle [2]. In fact David Griffiths will soon (Dec 4th) be giving an entire MIT Physics colloquium involving this very potential.

In any case, the solutions to (27) are writable in terms of Bessel functions:

$$\psi(z)=\sqrt{z}\left[C_1J_\gamma(i\beta z)+C_2Y_\gamma(i\beta z)\right] \tag{28}$$

were $\gamma=\frac{1}{2}\sqrt{1+4\alpha}$. Already we see that something peculiar will happen at the particular value of $\alpha=-1/4$. To find the spectrum, we would like to impose various conditions on the wavefunction–first, let us examine the large z behavior. The asymptotic behavior of the Bessel functions with imaginary argument is

$$J_\gamma(i\beta z)\sim\sqrt{\frac{2}{i\beta z}}\cosh(\beta z)\qquad Y_\gamma(i\beta z)\sim -i\sqrt{\frac{2}{i\beta z}}\sinh(\beta z) \tag{29}$$

where I have neglected some factors of $\pi/4$, etc. in the argument of the exponentials. It is clear that we need the linear combination appearing in ψ to decay exponentially at large z, which means that ψ takes the form

$$\psi(z)=C\sqrt{z}\left[J_\gamma(i\beta z)+iY_\gamma(i\beta z)\right] \tag{30}$$

Now we examine this at small z, where it becomes

$$\psi(z)=C\sqrt{z}\left[\frac{1}{\Gamma(\gamma+1)}\left(\frac{i\beta z}{2}\right)^\gamma-i\frac{\Gamma(\gamma)}{\pi}\left(\frac{2}{i\beta z}\right)^\gamma\right] \tag{31}$$

Now we finally need to worry about the true nature of $\gamma = \frac{1}{2}\sqrt{1+4\alpha}$. If $\alpha > -1/4$, then γ is real. Is this wave function then normalizable as we approach $z \to 0$? Actually, *yes*, it is; the relevant terms in $|\psi(z)|^2$ go like $z^{1+2\gamma}$, z, and $z^{1-2\gamma}$ respectively; these are actually all integrable at $z \to 0$ for sufficiently small γ, even if it is real. It appears that even if $\alpha > -\frac{1}{4}$ we can obtain *normalizable* negative energy bound states; furthermore we can get them for any value of β, i.e. a continuous spectrum of negative energy states. This can't be. Can we do better?

Yes we can. Recall our other boundary condition (20); evaluating this on two solutions (31) with different energies β_1, β_2 we obtain after some irritating algebra and manipulation of gamma functions the condition

$$\left(\frac{\beta_2}{\beta_1}\right)^{\gamma} + \left(\frac{\beta_1}{\beta_2}\right)^{\gamma} = 0 \tag{32}$$

This condition is never satisfied, regardless of the values of β_1, β_2 real. Thus we conclude that for real γ (i.e. $\alpha > -1/4$) we can never have a negative-energy bound state that satisfies the Hermiticity condition.

Finally, we move on to the case of interest, where $\alpha < -1/4$ and thus γ is imaginary; let's write it as $\gamma = ig$, where g is a real number. Our large z analysis goes through untouched, and thus (30) is still the correct form for the wavefunction. On the other hand, in the small-z analysis, z is now raised to an imaginary power and so is oscillatory $z^{ig} \sim \exp[ig\log(z)]$. After extensive use of Mathematica and manipulation of gamma functions that is too tedious to recount, you can show that at small z $\psi(z)$ is proportional to

$$\psi(z) \sim \sqrt{z}\cos\left[g\log\left(\frac{z\beta}{2}\right) + \delta\right] \tag{33}$$

δ is a g-dependent phase that I encourage you to calculate. This oscillates around 0 and so is probably normalizable at the boundary. Now the fun part; imposing the boundary condition (20) results in

$$g\left(\cos\left[g\log\left(\frac{z\beta_1}{2}\right)+\delta\right]\sin\left[g\log\left(\frac{z\beta_2}{2}\right)+\delta\right]\right) - 1 \leftrightarrow 2 = -g\sin[g(\log\beta_1 - \log\beta_2)] = 0 \tag{34}$$

This boundary condition can be solved, and the answer is that the (negative) energies β_1, β_2 must be related by

$$\beta_1 = \exp\left(\frac{n\pi}{g}\right)\beta_2, \qquad n \in \mathbb{Z} \tag{35}$$

Thus we have a discrete spectrum! Note that $\beta = 0$ is an accumulation point of the spectrum. And indeed, given some way to fix one of the energies in the spectrum, we can fix the entire set. However, I will not pursue this any further, and simply write down the punchline:

If $\alpha < -1/4$, we have normalizable negative-energy states on which the Schrodinger operator is Hermitian.

Now recall that a "negative-energy" state is actually an instability of the classical AdS wave equation, and $\alpha = \left(m^2 - \frac{1-d^2}{4}\right)$. Thus the criterion for the existence of a finite-(and conserved)-energy instability is simply

$$m^2 < -\frac{d^2}{4} \tag{36}$$

This is the Breitenlohner-Freedman bound. (Whew!)

III. DIMENSIONS OF VECTOR OPERATORS

Consider the bulk action

$$S = -\kappa\int d^{d+1}x\sqrt{g}\left(\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{m^2}{2}A^{\mu}A_{\mu}\right) \tag{37}$$

corresponding to a massive bulk vector field. The equations of motion from variation of this action are simply

$$\nabla_{\mu}F^{\mu\nu} = m^2A^{\nu} \tag{38}$$

which when written with normal partials reduces to

$$\frac{1}{\sqrt{-g}}\partial_{\mu}(\sqrt{-g}g^{\mu\nu}g^{\rho\sigma}F_{\nu\sigma}) - m^2A^{\rho} = 0 \tag{39}$$

For simplicity let us imagine that we are in Euclidean AdS and thus can put the full momentum in the t direction. We will examine polarizations that are not in the t or z directions, $\sigma \neq t, z$; in this case the equation of motion becomes after some manipulation

$$z^2\partial_z^2 A_\sigma + (3-d)z\partial_z A_\sigma + z^2\partial_t^2 A_\sigma - m^2 A_\sigma = 0 \tag{40}$$

Now if we substitute a power law solution $z^{\bar{\Delta}}$ into this equation, and examine the near-boundary behavior, we see that $\bar{\Delta}$ satisfies the equation

$$\bar{\Delta}(\bar{\Delta}+2-d) - m^2 = 0 \tag{41}$$

whole solutions are

$$\bar{\Delta}_\pm = \frac{1}{2}\left[(d-2) \pm \sqrt{(d-2)^2+4m^2}\right] \tag{42}$$

At this point we might remember the scalar field example and be tempted to simply say that $\bar{\Delta}_+$ is the answer and move on with our lives. This would be wrong, and the reason why is that A_μ has a vector index that appears a bit problematic; for example if we had worked instead with A^μ we would have obtained indices that were both shifted upwards by $+2$. Furthermore, we know that for $m = 0$ the bulk field is dual to a conserved current j^μ with dimension $d-1$; our formula does not reflect this. Of course the correct way to really find the scaling dimension is to do a slightly more extensive calculation and compute e.g. the correlator of the current j^μ.

A rather more heuristic way to do this is to work with an orthonormal basis for the A_μ's. This removes the ambiguity (the orthonormal basis components actually have some physical meaning, and $A^{\hat{\mu}} = A_{\hat{\mu}}$). $A_{\hat{\mu}} = zA_\mu$, and thus we should shift the indices both upwards by 1, giving us

$$\Delta_\pm = \frac{1}{2}\left[d \pm \sqrt{(d-2)^2+4m^2}\right] \tag{43}$$

Taking Δ_+ to be the conformal dimension of the boundary operator now gives us the correct result.

IV. SATURATING THE UNITARITY BOUND

We work with the normal massive scalar bulk action

$$S_{\text{usual}}[\phi] = -\frac{1}{2}\int d^{d+1}x\sqrt{g}\left[(\nabla\phi)^2 + m^2\phi^2\right] \tag{44}$$

and we would like to evaluate this action on a power-law field profile $\phi = z^\Delta e^{-ikx}$. This is not very difficult; in components the action is

$$S_{\text{usual}}[\phi] = -\frac{1}{2}\int d^{d+1}x\frac{1}{z^{d+1}}\left[z^2(\partial_z\phi)^2 + (m^2 - z^2k^2)\phi^2\right] \tag{45}$$

which on the power-law field becomes

$$S_{\text{usual}}[\phi] = -\frac{1}{2}\int d^{d+1}x(\Delta^2 + m^2 - z^2k^2)z^{2\Delta-(d+1)} \tag{46}$$

Peforming the integral over z, the leading divergence as $z \to 0$ is the term of the form

$$S_{\text{usual}}[\phi] \sim \lim_{z\to 0}\frac{\Delta^2+m^2}{2(2\Delta-d)}z^{2\Delta-d} \tag{47}$$

This will be finite only if $2\Delta - d > 0$. Now the conformal dimension of the operator dual to the bulk field is the exponent of the *normalizable* solution; thus it is clear that using this action the conformal dimension will always satisfy

$$\Delta > \frac{d}{2} \tag{48}$$

Somewhat more concretely, the two possible indices are

$$\Delta_{\pm} = \frac{1}{2}\left[d \pm \sqrt{d^2+4m^2}\right] \tag{49}$$

It is clear from here that only Δ_+ corresponds to a normalizable solution. However this state of affairs is not completely satisfactory, since the bound (48) on the dimension seems to be a bit too restrictive. To see how to get around it, consider the modified action

$$S_{KW}[\phi] = -\frac{1}{2}\int d^{d+1}\sqrt{g}\,\phi(-\Box+m^2)\phi \tag{50}$$

This action has precisely the same equations of motion as the previous one, yet it differs by a (divergent) boundary term and so will have a different numerical value. We now work out what this value is. On the AdS metric the action works out to be

$$S_{KW}[\phi] = -\frac{1}{2}\int d^{d+1}x\frac{1}{z^{d+1}}\phi\left(-z^{d+1}\partial_z(z^{2-(d+1)}\partial_z)+k^2z^2+m^2\right)\phi \tag{51}$$

and when acting on the power law solution a short computation results in

$$S_{KW}[\phi] = -\frac{1}{2}\int d^{d+1}xz^{2\Delta-(d+1)}\left([\Delta(d-\Delta)+m^2]+z^2k^2\right) \tag{52}$$

Now so far we have assumed nothing about Δ; let us now assume that Δ is one of the indices from (49), in which case the first and most divergent term in (52) vanishes. Nnote we could actually have guessed this immediately from the form of S_{KW}, as the power law with appropriate Δ is actually an exact solution to the equations of motion in the limit that $k \to 0$; thus the only nonzero piece in S_{KW} must be proportional to k^2. In any case, we see that the leading divergence in S_{KW} is

$$S_{KW} \sim \lim_{z\to 0}\frac{k^2}{2(2\Delta-d+2)}z^{2\Delta-d+2} \tag{53}$$

This is finite under the less restrictive condition that

$$\Delta > \frac{d-2}{2} \tag{54}$$

which is indeed the unitarity bound on the dimension of an operator in a CFT. Thus *this* lower bound actually makes sense. How can we achieve this? Note that if the mass satisfies

$$-\frac{d^2}{4} < m^2 < -\frac{d^2}{4}+1 \tag{55}$$

both $\Delta_\pm$ satisfy (54), and so either of them correspond to a normalizable solution and can be used as the dimension of the operator. Indeed, at the critical value $m^2 = 1-\frac{d^2}{4}$ we have $\Delta_- = (d-2)/2$ (Recall with the more restrictive bound (48) we could never even use any Δ_- for the dimension of the operator). More discussion along these lines can be found in [3].

---

[1] K. M. Case, "Singular Potentials," Phys. Rev. 80, 797 - 806 (1950)
[2] H. Hammer, B. Swingle, "On the limit cycle for the $1/r^2$ potential in momentum space," Annals Phys. 321 (2006) 306-317 [arXiv:quant-ph/0503074]
[3] I. Klebanov, E. Witten, "AdS/CFT Correspondence and Symmetry Breaking," Nucl.Phys. B556 (1999) 89-114. [arXiv:hep-th/9905104]