MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics
String Theory (8.821) – Prof. J. McGreevy – Fall 2008

Problem Set 1

Reading: §4 of d'Hoker-Freedman `http://arXiv.org/pdf/hep-th/0201253`

Due: Tuesday, September 23, 2008.

1. **Branes ending on branes.**

   The Dp-brane effective action contains a term of the form

   $$S \ni \int_{Dp} F \wedge C_{p-1},$$

   where C_{p-1} is the RR $p-1$ form, which couples minimally to D$(p-2)$-branes. Show that a D$(p-2)$ brane can end on a Dp brane without violating the Gauss law for the RR fields involved. Interpret the boundary of the D(p-2)-brane in terms of the worldvolume theory of the Dp brane. (If you like, focus on the case $p = 3$.)

2. **Timelike oscillators are evil.**

   Show that the commutation relation $[a, a^\dagger] = -1$ (which we found for the oscillators made from the time coordinates of the string) implies that either
   a) the energy $H = -a^\dagger a + E_0$[1] is unbounded below (if you treat $a^\dagger$ as the annihilation operator)
   or
   b) there are states with negative norms.

3. **Extremal Reissner-Nordstrom black hole.**

   As a warmup for the 10-d RR soliton, let's remind ourselves how the extremal RN black hole works.

   a) Consider Einstein-Maxwell theory in four dimensions, with action

   $$S_{EM} = \frac{1}{16\pi G_N} \int d^4x \sqrt{g} \left(\mathcal{R} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right)$$

[1] The -1 comes from $g^{00} = -1$.

Show that the Einstein equation $0 = \frac{\delta S_{EM}}{\delta g^{\mu\nu}}$ implies that

$$\mathcal{R}_{\mu\nu} = aG_N \left(2F_{\mu.}F_{\nu}^{.} - \frac{1}{2} g_{\mu\nu} F^2 \right)$$

for some constant a.

b) Consider the ansatz

$$ds^2 = H^{-2}(\rho) \left(-dt^2\right) + H^2(\rho) \left(d\rho^2 + \rho^2 d\Omega_2^2\right),$$
$$F = bdt \wedge d\left(H(\rho)^{-1}\right)$$

where b is some constant. Show that the Einstein equation $0 = \frac{\delta S_{EM}}{\delta g^{\mu\nu}}$ and Maxwell's equation $0 = \frac{\delta S_{EM}}{\delta A_\mu}$ are solved by the ansatz if H is a harmonic function on the $\mathbb{R}^3$ whose metric is

$$\gamma_{ab} dx^a dx^b := d\rho^2 + \rho^2 d\Omega_2^2.$$

Recall that H is harmonic iff $0 = \Box H = \frac{1}{\sqrt{\gamma}} \partial_a (\sqrt{\gamma} \gamma^{ab} \partial_b H)$.

c) Find the form of the harmonic function which gives a spherically symmetric solution; fix the two integration constants by demanding that i) the spacetime is asymptotically flat and ii) the black hole has charge Q, meaning $\int_{S^2 \text{ at fixed } \rho} \star F = Q$.

d) Take the near-horizon limit. Show that the geometry is $AdS_2 \times S^2$. Determine the relationship between the size of the throat and the charge of the hole.

[If you get stuck on this problem, see Appendix F of Kiritsis' book.]

d) If you're feeling brave, add some magnetic charge to the black hole. You will need to change the form of the gauge field to

$$F = bdt \wedge dH(\rho) + G(\rho)\Omega_2$$

where Ω_2 is the area 2-form on the sphere, and G is some function.

4. **RR soliton.**

In this problem we're going to check that the RR soliton is a solution of the equations of motion. The action for type IIB supergravity, when only the metric and the RR 5-form and possibly the dilaton are nontrivial can be written as

$$S_{IIB} = \frac{1}{16\pi G_N} \int d^{10}x \sqrt{g} \left(e^{-2\Phi} \left(\mathcal{R} + 4\partial_\mu \Phi \partial^\mu \Phi \right) - \frac{1}{5!} F^5_{.....} F^{5\ .....} + ... \right)$$

(The self-duality constraint $F^5 = \star F^5$ must be imposed as a constraint, and means that $dF^5 = 0$ implies the equations of motion for F^5.) By the way, this is the action for the *string frame* metric.

a) Show that the equations of motion from this action imply

$$\mathcal{R}_{\mu\nu} = a G_N e^{2\Phi} \left(5 F^5_{\mu....} F^{5}_{\nu}{}^{....} - \frac{1}{2} g_{\mu\nu} (F^5)^2 \right)$$

for some constant a.

b) Plug the following ansatz into the equations of motion:

$$ds^2 = \frac{1}{\sqrt{H(r)}} \eta_{\mu\nu} dx^\mu dx^\nu + \sqrt{H(r)} dy^2$$

$$F = b(1+\star) dt \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge dH^{-1}$$

$$\Phi = \phi_0$$

(b, ϕ_0 are constants.) Determine the constant b and the condition on the function H for this to solve the equations of motion.

To do this, there are two options – some kind of symbolic algebra program like Mathematica or Maple, or index-shuffling by hand. The latter is much more easily done using 'tetrad' or 'vielbein' methods. I always forget these and have to relearn them every time. For a lightning review of the vielbein method of computing curvatures, I recommend
d'Hoker-Freedman `http://arXiv.org/pdf/hep-th/0201253`, pages 100-101, or Argurio `http://arXiv.org/pdf/hep-th/9807171`, Appendix C. To help with the former option, I've posted an example curvature calculation in Mathematica on the pset webpage.

Note, by the way, that for values of p other than 3, the dilaton is not constant. With hindsight, this specialness of $p = 3$ is related to the fact that this is the critical dimension for YM theory, where g_{YM} is dimensionless.