8.821 F2008 Lecture 12:
Boundary of AdS; Poincaré patch; wave equation in AdS

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Today:
1. the boundary of AdS
2. Poincaré patch
3. motivate boundary value problem
4. wave equation in AdS.

1 The boundary of AdS

We defined the Lorentzian AdS_{p+2} as the locus $\{\eta_{ab}X^aX^b = -L^2\} \subset \mathbb{R}^{p+1,2}$, where

$$\eta_{ab}X^aX^b = -X_0^2 + \sum_{i=1}^{p+1} X_i^2 - X_{p+2}^2 = -L^2 \tag{1}$$

The metric is

$$ds^2_{AdS} = \eta_{ab}X^aX^b|_{(1)} = L^2\left[-\cosh^2\rho\, d\tau^2 + d\rho^2 + \sinh^2\rho\, d\Omega_p^2\right] \tag{2}$$

1.1 Projective boundary

Take a solution $V = \left(X_0, \vec{X}, X_{p+2}\right)$ of equation (1). Reach the boundary by rescaling X, preserving (1). Let $X = \lambda\tilde{X}$, then equation (1) becomes

$$\eta_{ab}\tilde{X}^a\tilde{X}^b = -\frac{L^2}{\lambda^2} \tag{3}$$

We now take $\lambda \to \infty$, the boundary is

$$\{\eta_{ab}\tilde{X}^a\tilde{X}^b = 0\} \;/\; \{\tilde{X} \sim \lambda\tilde{X}\} \simeq \mathbb{R}^{p,1} \tag{4}$$

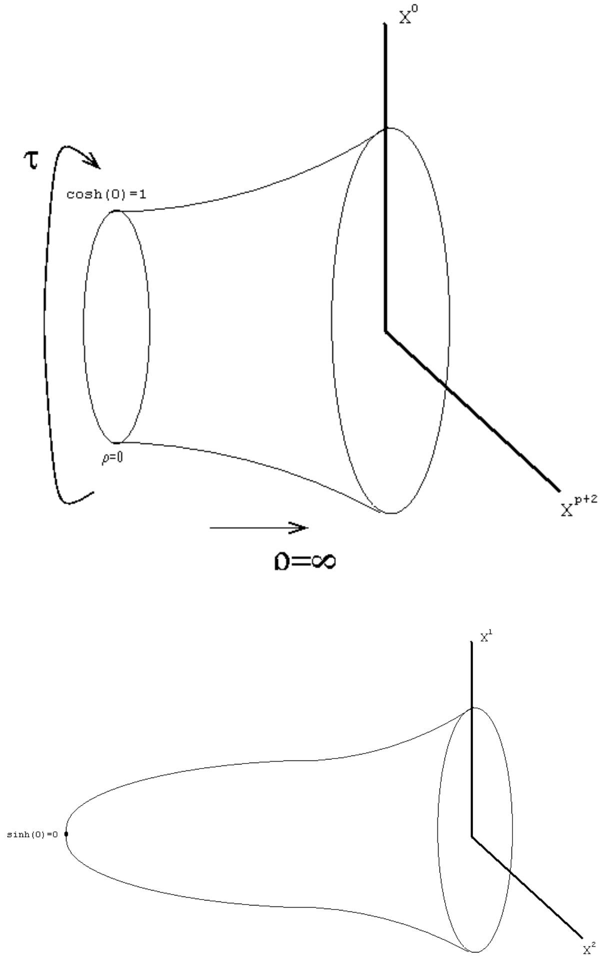

Figure 1: Lorentzian AdS: The left-right axis is the ρ direction. At $\rho = 0$, the S^p in the lower figure shrinks to zero size (like $\sinh \rho$), while the radius of the τ direction, depicted in the top figure, approaches a constant (like $\cosh \rho$).

This relation can also be read as follows: the boundary of AdS is the set of lightrays in $\mathbb{R}^{p+1,2}$, modulo the rescaling. Recall that this is exactly parametrized by points in $\mathbb{R}^{p,1}$ as:

$$\rho^a = \kappa \left(X^\mu, \frac{1}{2}(1 - X^2), \frac{1}{2}(1 + X^2) \right). \tag{5}$$

We used this fact earlier to make write the $SO(p+1,2)$ action of the conformal group on $\mathbb{R}^{p,1}$ in a linear way. The fact that the conformal group of $\mathbb{R}^{p,1}$ has a nice action on the boundary of AdS is very encouraging.

Alternative decomposition I

Fix λ by imposing $1 = \vec{X}^2 = \sum_{i=1}^{p+1} X_i^2$. Then we have

$$X_0^2 + X_{p+2}^2 = \vec{X}^2 = 1 \qquad \Rightarrow \qquad \partial AdS = S^1 \times S^p \tag{6}$$

Alternative decomposition II

Let $u_\pm = X_0 \pm iX_{p+1}$. Then (1) $\Rightarrow -u_+u_- + \vec{X}^2 = 0$.
If $u_+ \neq 0$ set $u_+ = 1 \quad \Rightarrow \quad u_- = \vec{X}^2$
If $u_- \neq 0$ set $u_- = 1 \quad \Rightarrow \quad u_+ = \vec{X}^2$
Then $\tilde{\vec{X}} = \frac{\vec{X}}{\vec{X}^2}$. u_- is 'the point at ∞'. The boundary is compact.

1.2 Penrose diagram (one more description of the boundary)

Let $d\Theta = \frac{d\rho}{\cosh\rho}$ (this variable was called 'squiggle' in lecture). The metric in these new coordinates results in

$$ds^2 = \cosh^2\rho \left[-d\tau^2 + d\Theta^2 + \tan^2\frac{\Theta}{2}\, d\Omega_p^2\right] \tag{7}$$

and therefore

$$\tan\frac{\Theta}{2} = \tanh\frac{\rho}{2} \qquad\qquad \Theta \in [0, \pi/2] \tag{8}$$

The boundary is $\{\Theta = \pi/2\} \sim \mathbb{R} \times S^p$. Note that the metric on the boundary is only specified up

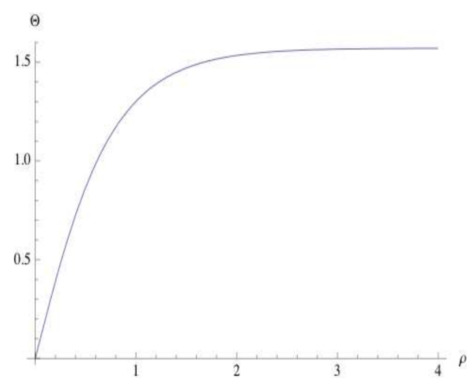

Figure 2: The squiggle variable Θ runs from 0 to $\pi/2$ as ρ goes from 0 to ∞

to rescaling, *i.e.* a Weyl transformation.

But why do we care about this boundary more than say the conformal boundary of Minkowski space? The answer is in the next two subsections.

1.3 Massless geodesics

The massless geodesics are given by the condition $ds^2 = 0$, which implies

$$0 = ds^2 = L^2 \left(-\cosh^2\rho\, d\tau^2 + d\rho^2\right) \quad \Rightarrow \quad \cosh\rho = \frac{d\rho}{d\tau} \quad \Rightarrow \quad d\tau = \frac{d\rho}{\cosh\rho} = d\Theta \tag{9}$$

Θ is the time elapsed for a static observer. Whether the lightray reflects off the boundary depends on the BC's. Hence: Cauchy problem problem.

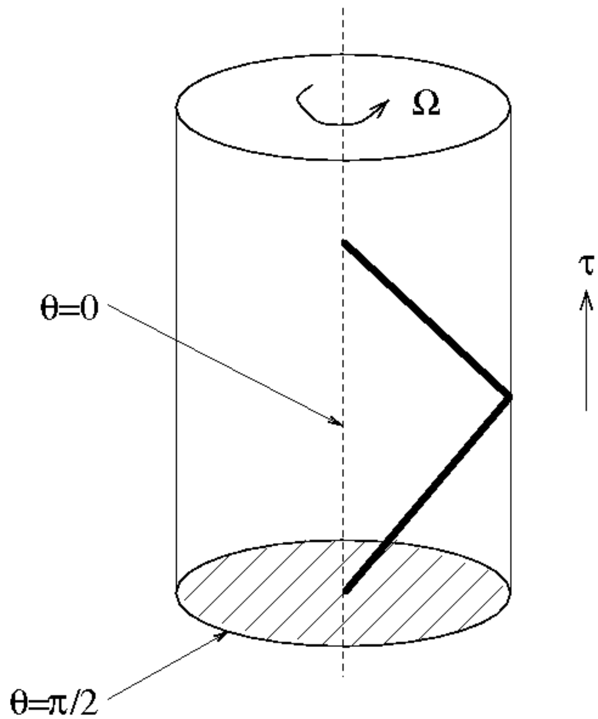

Figure 3: Massless geodesics

1.4 Massive geodesics

The action for a massive relativistic point particle is

$$S = m \int ds = m \int \sqrt{g_{\mu\nu}\dot{X}^\mu \dot{X}^\nu} \qquad \dot{X}^\mu = \partial_\tau X^\mu \tag{10}$$

The equation of motion is

$$\frac{\delta S}{\delta X_\mu} = 0 \quad \Rightarrow \quad \ddot{X}^\mu + \Gamma^\mu_{\nu\lambda}\dot{X}^\nu \dot{X}^\lambda = 0 \tag{11}$$

where the second equation follows if $\dot{X} = \partial_s X$ where s is proper time. If we assume $\dot{\Omega} = 0$ the action is

$$S = mL \int d\tau \sqrt{\cosh^2 \rho - (\partial_\tau \rho)^2}.$$

You will show on problem set 3 that this has an oscillatory solution around $\rho = 0$, it never reaches ∞.

2 Poincaré patch

Pick out X^{p+1} from among the X^i. This will break the $SO(p+1)$ symmetry of the p-sphere. Let

$$\begin{cases} X^\mu = \frac{L}{z} x^\mu \\ X_{p+2} + X_{p+1} = \frac{L}{z} \\ -X_{p+2} + X_{p+1} = v \end{cases} \tag{12}$$

Equation (1) and the metric become

$$\begin{aligned} \frac{L}{\tau} v - \frac{L^2}{z^2} x^\mu x_\mu &= -L^2 \\ ds^2 &= L^2 \frac{dz^2 + dx^\mu \, dx_\mu}{z^2} \end{aligned} \tag{13}$$

(same cancellation as UHP). This is the metric which we showed has

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \Lambda g_{\mu\nu} \qquad\qquad \Lambda = -\frac{(p+1)(p+2)}{2L^2} \tag{14}$$

NOTE: it covers part of AdS. As $z \to \infty$, $\partial/\partial t$ becomes NULL (Poincaré horizon).

CLAIM: relation between Poincaré patch and global time is state-operator correspondence.

EVIDENCE: symmetries $\to SO(p,1) \times \mathbb{R}^{p+1}$ and $SO(p+1) \times SO(2)$.

2.1 Towards CFT correlators from fields in AdS

Our goal is to evaluate $\langle e^{-\int \phi_0 \, \mathcal{O}} \rangle_{CFT} \equiv e^{-W_{CFT}[\phi_0]}$.
Conjecture: $\langle e^{-\int \phi_0 \, \mathcal{O}} \rangle_{CFT} = Z_{strings\, in\, AdS}[\phi_0]$, but we cannot compute it. The pratical version is the following

$$W_{CFT}[\phi_0] = -\ln \langle e^{\int \phi_0 \mathcal{O}} \rangle_{CFT} \simeq \text{extremum}_{\phi|_{z=\epsilon} = \phi_0} \left(N^2 I_{SUGRA}[\phi] \right) + O\left(\frac{1}{N^2}\right) + O\left(\frac{1}{\sqrt{\lambda}}\right) \tag{15}$$

A few comments:

- The supergravity description is valid for large N and large λ. In (15) we've made the N-dependence explicit: in units of the AdS radius, the Newton constant is $\frac{1}{G_N} = N^2$. I_{SUGRA} is some dimensionless action.

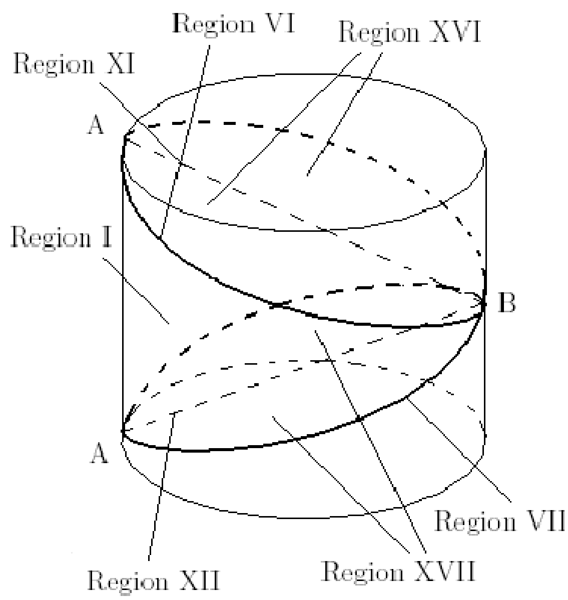

Figure 4: Poincaré patch

- anticipating divergences at $z \to 0$, we introduce a cutoff (which will be a UV cutoff in the CFT) and set boundary conditions at $z = \epsilon$.
- Eqn (15) is written as if there is just one field in the bulk. Really there is a ϕ for every operator $\mathcal{O}$ in the dual field theory.

We'll say 'ϕ couples to $\mathcal{O}$' at the boundary. How to match? We give four examples

1. Dilaton field.
   Before near horizon limit, we have D3-branes in $\mathbb{R}^{10}$; the asymptotic value of the dilaton determines the string coupling constant $g_s = \langle e^{\phi(x \to \infty)} \rangle$. The YM coupling on D3's is $g_{YM}^2 = g_s$.
   Changing $\phi \to \phi + \delta\phi$ we get
   $$\delta S = \int \frac{\delta\phi}{g_s^2} Tr\left[F^2 + \ldots\right] \tag{16}$$
   where the dots stand for all the CP-even term in the lagrangian. In conclusion we have
   $$Z_{strings}\left[\phi \to \phi + \delta\phi\right] \simeq \langle e^{\frac{1}{g_s^2}\int \delta\phi Tr[F^2]} \rangle_{CFT} \tag{17}$$
   The dilaton couples to all the terms in the lagrangian which are CP invariant.
2. RR axion.
   We have that $\tau_{str} = \frac{i}{g_s} + \frac{\chi}{2\pi}$ tranforms under $SL(2, \mathbb{C})$ nicely, like $\tau = \frac{i}{g_s} + \frac{\theta}{2\pi}$. Therefore
   $$\chi \leftrightarrow Tr[F \wedge F] \tag{18}$$
   This time CP-odd terms

3. Stress energy tensor.
   The tensor $T_{\mu\nu}$ is the response of a local QFT to local change in the metric. $S_{QFT} \supset \int \gamma_{\mu\nu} T^{\mu\nu}$. Here we are writing $\gamma_{\mu\nu}$ for the metric on the boundary. In this case

$$g_{\mu\nu} \leftrightarrow T_{\mu\nu} \tag{19}$$

4. IIB in $\mathrm{AdS}_5 \times S^5$.
   Isometry on $S^5 \to SO(6)$ Kaluza-Klein (KK) gauge fields $\leftrightarrow SO(6)_R = SU(4)_R$. In this case the correspondence is between these gauge fields and the R-current operators

$$A_\mu^{KK\,a} \leftrightarrow J_R^{\mu\,a} \tag{20}$$

   i.e. $S_{bdy} \ni \int A_\mu^a J_a^\mu$

2.2 Useful visualization

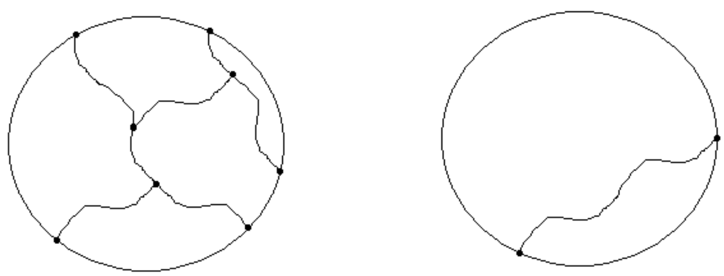

Figure 5: Feynman graphs in AdS. We do the one with two ext. legs first

Classical field theory in bulk (boundary value problem).
Extr. of classical action (expand about quadratic solution in powers of ϕ_0) = tree level SUGRA Feynman graphs.
BUT: usually (QFT in $\mathbb{R}^{D,1}$), ext. legs of graphs = wavefunction of asymptotic states (example: plane waves).
In AdS: ext. legs of graphs determined by boundary behavior of ϕ ('bulk-to-boundary propagators').

3 Wave equation in AdS

We work in Poincaré coordinates. The metric is

$$ds^2 = L^2 \frac{dz^2 + dx^\mu\, dx_\mu}{z^2} \equiv g_{AB} dz^A dz^B \qquad A = 0, \ldots, p+1 \tag{21}$$

The action for a scalar field is

$$S = -\frac{\eta}{2} \int d^{p+2}x \sqrt{g} \left[g^{AB} \partial_A \phi \partial_B \phi + \frac{1}{2} m^2 \phi^2 + b\phi^3 + \ldots \right] \tag{22}$$

For this metric $\sqrt{g} = \sqrt{|det\, g|} = \left(\frac{L}{z}\right)^2$.
Since ϕ is a scalar field we can rewrite the kinetic term as

$$g^{AB}\partial_A\phi\partial_B\phi = (\partial\phi)^2 = g^{AB}D_A\phi D_B\phi \tag{23}$$

where D_A is the covariant derivative. Thus we can use $D_A(g_{BC}) = 0$ to move the Ds around the gs with impunity. By integrating by parts we can rewrite the action as

$$S = -\frac{\eta}{2}\int d^{p+2}x\left[\partial_A\left(\sqrt{g}\,g^{AB}\phi\,\partial_B\phi\right) - \phi\,\partial_A\left(\sqrt{g}\,g^{AB}\,\partial_B\phi\right) + \sqrt{g}\left(m^2\phi^2 + \ldots\right)\right] \tag{24}$$

and finally by using the Stokes theorem we can rewrite the action as

$$S = -\frac{\eta}{2}\int_{\partial AdS} d^{p+1}x\,\sqrt{g}\,g^{zB}\phi\,\partial_B\phi - \frac{\eta}{2}\int\sqrt{g}\,\phi\left(-\Box + m^2\right)\phi + O(\phi^3) \tag{25}$$

where we define $\Box\phi = \frac{1}{\sqrt{g}}\partial_A\left(\sqrt{g}\,g^{AB}\,\partial_B\right)\phi = D^A D_A\phi$.

We can rewrite it more covariantly as

$$\int_{\mathcal{M}}\sqrt{g}\,D_A J^A = \int_{\partial\mathcal{M}}\sqrt{\gamma}\,n_A J^A \tag{26}$$

The metric tensor γ is defined as

$$ds^2|_{z=\epsilon} \equiv \gamma_{\mu\nu}dx^\mu dx^\nu = \frac{L^2}{\epsilon^2}\eta_{\mu\nu}dx^\mu dx^\nu \tag{27}$$

i.e. it is the induced metric on the boundary surface $z = \epsilon$. The vector n_A is a unit vector normal to boundary ($z = \epsilon$). We can find an expression for it

$$n_A \propto \frac{\partial}{\partial z} \quad g_{AB}n^A n^B|_{z=\epsilon} = 1 \qquad \Rightarrow \qquad n = \frac{1}{\sqrt{g_{zz}}}\frac{\partial}{\partial z} = \frac{z}{L}\frac{\partial}{\partial z} \tag{28}$$