8.821 F2008 Lecture 11: CFT continued; geometry of AdS

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In this session, we are going to talk about the following topics.

1. We are making a few comments about CFT.

2. We are discussing spheres and hyperboloids.

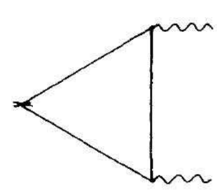
3. Finally we are focusing on Lorentzian AdS and its boundary.

1 Conformal Symmetry

1.1 Weyl anomaly

Quantumly, conformal symmetry in a curved space (with even number of dimensions) could be anomalous, that is $ds^2 \to \Omega(x)ds^2$ could be no longer a symmetry of the full quantum theory. This anomaly can be evaluated from the following diagram with operator T^μ_μ inserted at the left vertex.

Figure 1: A contribution to the Weyl anomaly.



The conformal anomaly signals a nonzero value for the trace of the energy-momentum tensor. In a curved spacetime, it is related to the curvature:

$$T^\mu_\mu \sim \mathcal{R}^{D/2}$$

where $\mathcal{R}$ denotes some scalar contractions of curvature tensors and D is the number of spacetime dimensions; the power is determined by dimensional analysis.

For the special case of $D = 2$, this is

$$T^\mu_\mu = -\frac{c}{12}\mathcal{R}^{(2)} \tag{1}$$

where $\mathcal{R}^{(2)}$ is the Ricci scalar in two dimensions and c is the central charge of the Virasoro algebra of the 2d CFT. Also in $D = 4$, the anomaly is given by $T^\mu_\mu = aW + cGB$ where W and B are defined as

$$W = (\text{Weyl tensor})^2 = R^{\cdots\cdot}R_{\cdots\cdot} - 2R^{\cdot\cdot}R_{\cdot\cdot} + \frac{1}{3}R^2 \tag{2}$$

$$GB = \text{Euler density} = R^{\cdots\cdot}R_{\cdots\cdot} - 4R^{\cdot\cdot}R_{\cdot\cdot} + R^2; \tag{3}$$

The Gauss-Bonnet tensor is the 'Euler density' in the sense that

$$\int_{\mathcal{M}} GB = \chi(\mathcal{M}) = \sum(-1)^p b_p(\mathcal{M})$$

where $\chi(\mathcal{M})$ is the Euler character of $\mathcal{M}$. c and a are "central charges" which are proportional to the number of degrees of freedom of CFT.

In $D = 2$ the central charge can be defined even away from critical theories as follows:

$$c = \lim_{z\to 0} \quad z^4\langle T(z)T(0)\rangle. \tag{4}$$

The Zamolodchikov c-theorem says that this quantity is equal to the c defined above when evaluated at a RG fixed point, and is monotonically decreasing under RG flow. In four dimensions, a longstanding conjecture of Cardy suggests that a should decrease under RG flow, but there are now some counterexamples [1].

1.2 OPE

Any local QFT has an Operator Product Exansion (OPE). The idea is that any local disturbance is created by local operators. Consider some correlation function of local operators $\langle\prod \mathcal{O}_i(x_i)\rangle$; focus on one local operator $\mathcal{O}_1(x)$ inserted at point x and there is another one $\mathcal{O}_2(y)$ at y. Consider an imaginary line around them and suppose that $|x-y| < \epsilon$ which is their distance from nearby operators. Then we can squint the local disturbance by a superposition of local operators

$$\mathcal{O}_1(x_1)\mathcal{O}_2(x_2) = \sum_{n,\text{ all operators with same quantum numbers}} c^n_{12}\mathcal{O}_n(y) \tag{5}$$

The coefficients are independent of the other operators in the correlator. The coefficients are in general hard to know. In a CFT, we can plug (5) into a three-point function and we find that the OPE coefficients are determined to be

$$c^n_{12}(x-y) = \frac{c^n_{12}}{|x-y|^{\Delta_1+\Delta_2-\Delta_n}} \tag{6}$$

where c^n_{12} are proportional to the 3-point interaction coefficient.

[1] Shapere and Tachikawa, 0809.3238

1.3 Conformal Dimensions

Let's give a few examples of operators with definite conformal dimension:

a) Energy-momentum tensor: The conformal dimension is $\Delta = D$ which is guaranteed by dimensional analysis. The energy-momentum tensor must be coupled to $g^{\mu\nu}$ by definition, and the metric components are dimensionless. There are other more algebraic arguments for this (use the Conformal Ward identities to constrain the OPE of the stress tensor with itself).

b) For a global symmetry, there is a conserved current j^μ which has $\Delta = D-1$. The current can be coupled to a gauge field in which case these is a gauge invariance that guarantees the conservation of the current, that is $\partial^\mu j_\mu = 0$, and fixes the dimension.

In general, the conformal dimensions are very hard to know, however there are some lower bounds on them in *unitary* CFTs. The lower bound constrains the possible values of the conformal dimension according to their spins. A few examples of this is as follows (for a full analysis see hep-th/9712074)

$$\Delta_{\text{scalar operator}} \geq \frac{D-2}{2} \qquad = \text{free field dimension}$$
$$\Delta_{\text{spin 1/2 operaor}} \geq \frac{D-1}{2} \qquad = \text{free field dimension}$$
$$\Delta_{\text{spin 1}} \geq D-1 \qquad = \text{conserved current dimension}$$

In the last equation, spin 1 is meant to be in $(\frac{1}{2},\frac{1}{2})$ representation as opposed to $(1,0)$.

1.4 Thermodynamics of a CFT

As a last remark on CFT we discuss the thermodynamics of a CFT. The partition function is defined as

$$Z_{CFT} = \text{Tr}_{CFT}\left(\exp(-H/T)\right).$$

In the thermodynamic limit, $\ln Z$ is proportional to the volume of the space. $\ln Z$ is a dimensionless quantity. Hence, we must have $\ln Z \sim VT^d$ (d is the number of spatial dimensions) in the absence of any other energy scales (such as a chemical potential for some conserved charge). The free energy then will be

$$F = -T\ln Z = cVT^{d+1}.$$

where this c is also somehow proportional to the number of degrees of freedom of CFT.

Exploiting simple facts about CFT we can derive some interesting results. We must regard $T^\mu_\mu = 0$ as an operator equation in the full quantum theory. So we might be tempted to put it inside $\text{Tr}(e^{-H/T})$. The operator equation then translates into the following equation

$$0 = \text{Tr}(T^\mu_\mu e^{-H/T}) = \langle T_{00}\rangle - \langle T_{ii}\rangle = \mathcal{E} - dP.$$

This last relation gives the speed of the sound

$$c_s = \sqrt{\left(\frac{\partial P}{\partial \mathcal{E}}\right)_S} = \sqrt{\frac{1}{d}} \tag{7}$$

2 Sphere, Hyperboloid and AdS

The *AdS* space has a constant negative curvature. It is actually the most symmetrical space with a negative curvature. The most symmetric (Euclidean) space with a positive curvature is obviously a sphere. A useful and immediate description of spheres and their metrics arises by embedding in a higher dimensional space. Below we will use the same logic to investigate the *AdS* space, but as a warmup we start with a sphere

$$S^d = \{\sum_{i=1}^{d+1} x_i^2 = L^2\} \subset \mathbb{R}^n \tag{8}$$

with the flat metric $ds^2 = \sum_{i=1}^{d+1} dx_i^2$. Note that the defining equation of the manifold respects the symmetries of the ambient space. That is, under the transformation $x_i \to \Lambda_i{}^j x_j$ for $\Lambda \subset SO(d+1)$, the manifold will be mapped to itself. In other words, the embedding is "isometric".

We can solve equation (8) to find a set of global coordinates. In two dimensions, for example, we find the familiar spherical coordinates

$$\begin{aligned} x_1 &= L\cos\theta\cos\phi, \\ x_2 &= L\cos\theta\sin\phi, \\ x_3 &= L\sin\theta \end{aligned}$$

The metric then would look like

$$ds^2_{S^2} = L^2(d\theta^2 + \cos^2\theta d\phi^2).$$

2.1 Euclidean AdS = hyperbolic space

Our next goal is to describe the (Euclidean) hyperbolic space in a higher dimensional space. In three dimensions this is like the familiar hyperboloid defined as

$$\{x^2 - y^2 - z^2 = R^2\} \subset \mathbb{R}^3, \qquad ds^2 = dx^2 + dy^2 + dz^2 \tag{9}$$

However, it will more useful to embed hyberbolic space in *Minkowski* space; this is because the locus (9) doesn't respect the $SO(3)$ symmetries of the ambient $\mathbb{R}^3$ metric. So instead, we define d-dimensional hyperbolic space to be the locus

$$H_d = \{-X_{d+1}^2 + \sum_{i=1}^{d} X_i^2 = -L^2\} \subset \mathbb{R}^{d,1} \tag{10}$$

with the metric given by

$$ds^2 = -dX_{d+1}^2 + \sum_{i=1}^{d} dX_i^2$$

The manifold defined by equation (10) has two disconnected branches with $X_{d+1} \geq 1$ and $X_{d+1} \leq -1$. Here we will only consider the connected subspace with $X_{d+1} \geq 1$. We can easily see that this space is spacelike. The argument goes as follows. Define vector $v^A = (x_{d+1}, \vec{x})$. This is a timelike vector for any $x \in H_d$, for $(v^A)^2 = -L^2$. suppose that δv^A is the tangent vector to H_d at $(x_{d+1}, \vec{x})$. Therefor we have

$$(v^A + \delta v^A)^2 = -L^2 \Leftrightarrow v^A.\delta v^A = 0$$

which means that δv^A is spacelike (otherwise we have two timelike orthogonal vectors).

Note that the induced metric on H_d will manifestly preserve the $SO(d,1)$ symmetry of the ambient Minkowski space. This space is actually homogenous, in the sense that any point can be mapped to any other by an $SO(d,1)$ transformation.

In this case, we can choose the set of global coordinates as

$$x_{d+1} = L\cosh\rho,$$
$$x_i = L\sinh\rho \;\; \Omega_i, \qquad \sum_{i=1}^{d} {\Omega_i}^2 = 1.$$

Using $\sum {\Omega_i}^2 = 1$ and thus $\Omega_i d\Omega_i = 0$, we find the induced metric on the hyperbolic space

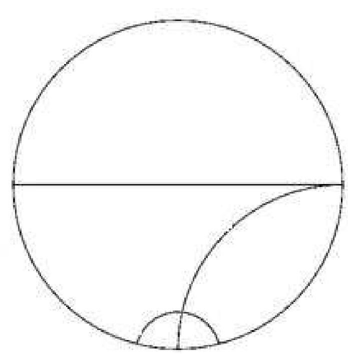
$$ds^2 = L^2(d\rho^2 + \sinh^2\rho \sum d{\Omega_i}^2) = L^2(d\rho^2 + \sinh^2\rho d\Omega_p^2)$$

where $d\Omega_p^2$ is the round metric on the unit p-sphere.

We can also define the following coordinates which conformally compactify the space (*i.e.* map the infinite coordinate range of ρ to a finite range in the new coordinate). Define $d\Theta = \frac{d\rho}{\sinh\rho}$ from which we find that $\tan(\Theta/2) = \tanh(\rho/2)$. The metric then would take the following form

$$ds^2 = L^2(d\Theta^2 + \tan^2(\Theta/2)d\Omega^2)(\cosh^2\rho)$$

Figure 2: Poincare disk.



This space is called Poincaré disk and it is topologically a ball whose boundary is $\partial H = S^{d-1}$. The distance from some point in the interior to the boundary is infinite.

Let's also define the following subset of the manifold

$$S(\bar{\rho}) = \{p \in H, \rho(p) < \bar{\rho}\}.$$

The curious fact is that

$$\lim_{\rho\to\infty} \frac{\text{Area}(S(\rho))}{\text{Volume}(S(\rho))} \sim \frac{1}{L}.$$

We can also define the following coordinates (UHP coordinates). Let's define

$$u = X_d - X_{d+1}, \qquad v = X_d + X_{d+1}.$$

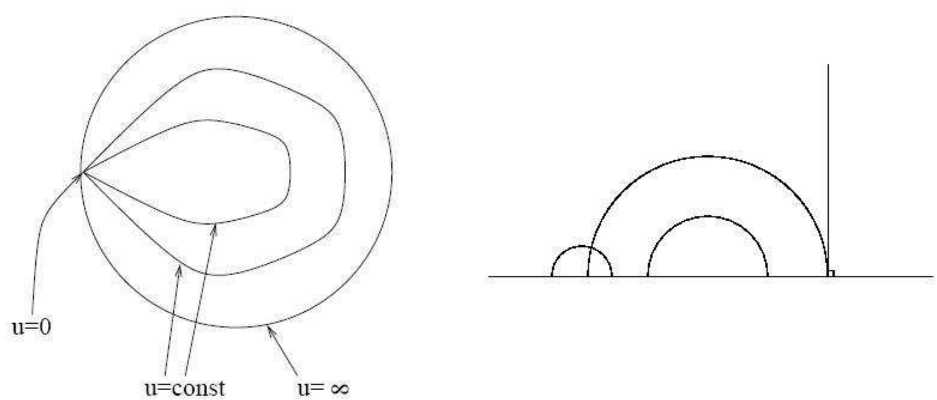
The equation (10) then becomes

$$-L^2 = -X_{d+1}^2 + X_d^2 + \sum_{i=1}^{d-1} X_i^2 = uv + \sum \vec{X}^2,$$

The ambient metric also becomes

$$ds^2 = dudv + d\vec{X}^2.$$

Figure 3: The Poincaré plane in the global coordinates and in the Upper Half Plane coordinates.



We can now solve for a variable, say v, in the previous equations. Let's also redefine $\vec{X}$ as $\vec{x} = \frac{\vec{X}}{u}L$. The metric in these new coordinates is

$$\begin{aligned} ds^2|_{H_d} &= dudv + d\vec{x}^2 = L^2\frac{du^2}{u^2} + \frac{u^2 d\vec{x}^2}{L^2} && (11) \\ &= L^2\left(\frac{dz^2 + d\vec{x}^2}{z^2}\right) && (12) \end{aligned}$$

where in the last line, we introduced the new variable $z = R^2/u$. Note that $z > 0$. These last coordinates define the Poincaré Upper Half Plane (UHP).

2.2 Lorentzian AdS

Now we are in a position to focus on the problem which is really of our interest. This is the Lorentzian AdS_{p+2}. Let's first define the signature that we will need in this case. The embedding space is $R^{p+1,2}$ with metric $\eta_{ab} = \text{diag}(-\underbrace{++\cdots+}_{p+1's}-)$. The metric is then

$$ds^2 = -dx_0^2 + \underbrace{d\vec{x}^2}_{\text{a } p+1 \text{ vector}} - dx_{p+2}^2 \quad (two \ \ times!)$$

The AdS space is then defined as

$$\{\eta_{ab}X^a X^b = -L^2\} \subset \mathbb{R}^{p+1,2} \tag{13}$$

Despite the appearance of two times in the definitions, this space has really only one time direction. The argument is the same as above. (Note that the vector orthogonal to the AdS in $\mathbb{R}^{p+1,2}$ is timelike).

The symmetries of the AdS space is apparent from equation (13). It has a $SO(p+1,2)$ symmetry and it is also homogenous. Let's define the following set of coordinates (called *global coordinates*)

$$\begin{aligned} X_0 &= L\cosh\rho\cos\tau, \\ X_i &= L\sinh\rho \quad \Omega_i, \\ X_{p+2} &= L\cosh\rho\sin\tau \end{aligned}$$

where $\sum_{i=1}^{p+1}\Omega_i^2 = 1$. In these coordinates, the metric becomes

$$ds^2 = L^2[-\cosh^2\rho d\tau^2 + d\rho^2 + \sinh^2\rho \ \ \underbrace{d\Omega_p^2}_{SO(p+1)} \ ]$$

This metric has a visible $SO(2)$ isometry under which τ goes to $\tau + a$. However τ is supposed to be the time direction. Near $\rho = 0$, for example, the spacetime looks like

$$\underbrace{S^1}_{timelike} \times \mathbb{R}^{p+1}$$

So in order to avoid closed timelike curves and to make any sense of the theory, we should *unwrap* τ, that is, we should instead use the covering space of the manifold that doesn't identify τ and $\tau + 2\pi$.

Figure 4: AdS_{p+2} is realized as a hypeboloid in $\mathbb{R}^{p+1,2}$