

8.821 F2008 Lecture 10: CFTs in $D > 2$

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In today's lecture we'll elaborate on what conformal invariance really means for a QFT. Remember, we'll keep working with CFTs in greater than 2 spacetime dimensions.

A very good reference is *Conformal Field Theory* by Di Francesco et al, p95-107.

1 Conformal Transformations

To begin, we'll give a few different perspectives on what conformal invariance is and what conformal transformations are. Our first perspective was emphasized by Brian Swingle after last lecture.

Recall from last lecture that the *group structure* of the conformal group is the set of coordinate transformations:

$$\begin{aligned}x &\rightarrow x' \\ g'_{\mu\nu}(x') &= \Omega(x)g_{\mu\nu}(x)\end{aligned}\tag{1}$$

However, we should point out that coordinate transformations don't do anything at all in the sense that the metric is invariant, $ds'^2 = ds^2$, and all we did was relabel points. To say something useful about physics, we really want to compare a theory at *different* distances. For example, we would like to compare the correlators of physical observables separated at different distances. Conformal symmetry relates these kinds of observables!

To implement a conformal transformation, we need to change the metric via a Weyl rescaling

$$\begin{aligned}x &\rightarrow x \\ g_{\mu\nu} &\rightarrow \Omega(x)g_{\mu\nu}\end{aligned}\tag{2}$$

which changes physical distances $ds^2 \rightarrow \Omega(x)ds^2$. Now follow this by a coordinate transformation such as the one in Eq.(1), so that $x \rightarrow x'$ and $\Omega(x)g_{\mu\nu} \rightarrow g'_{\mu\nu}(x')$. This takes us to a coordinate system where the metric has the same form as the one we started with, but the points have all been moved around and pushed closer together or farther apart. Therefore, we can view the conformal group as those coordinate transformations which can 'undo' a Weyl rescaling.

1.1 Scale transformations

Another useful perspective on conformal invariance, particularly for AdS/CFT, is to think about scale transformations.

Any local (relativistic) QFT has a stress-energy tensor $T^{\mu\nu}$ which is conserved if the theory is translation invariant. $T^{\mu\nu}$ encodes the linear response of the action to a small change in the metric:

$$\delta S = \int T^{\mu\nu} \delta g_{\mu\nu} \quad (3)$$

If we put in gravity, then $T^{\mu\nu} = 0$ by the equations of motion. But let's return to QFT without gravity. Consider making a *scale transformation*, which changes the metric

$$\delta g_{\mu\nu} = 2\lambda g_{\mu\nu} \quad \rightarrow \quad \delta S = \int T_{\mu}^{\mu} 2\lambda \quad (4)$$

where λ is a constant. Therefore we see that if $T_{\mu}^{\mu} = 0$ then the theory is scale invariant.

In fact, since nothing we said above depended on the fact that λ is a constant, and since our theory is a local theory, we can actually make the following transformation:

$$\delta g_{\mu\nu} = \delta\Omega(x) g_{\mu\nu} \quad \rightarrow \quad \delta S = \int T_{\mu}^{\mu} \delta\Omega(x) \quad (5)$$

We conclude from the two equations above that, at least classically,

- If $T_{\mu}^{\mu} = 0$ the theory has both scale invariance and conformal invariance.
- If the theory is conformally invariant, it is also scale invariant, and $T_{\mu}^{\mu} = 0$

However scale invariance does not quite imply conformal invariance. For more details please see Polchinski's paper, *Scale and conformal invariance in quantum field theory*.

The conserved currents and charges of the transformations above are:

$$S_{\mu} = x^{\nu} T_{\mu\nu} \quad \rightarrow \quad D \equiv \int S_0 d^d x \quad (6)$$

$$C_{\mu\nu} = (2x_{\mu} x_{\nu} - x^2 g_{\mu\nu}) T_{\nu}^{\lambda} \quad \rightarrow \quad C_{\mu} \equiv \int C_{0\mu} d^d x \quad (7)$$

since both $\partial^{\mu} S_{\mu}$ and $\partial^{\mu} C_{\mu\nu}$ are proportional to T_{μ}^{μ} .

1.2 Geometric Interpretation

Lastly we'd like to give an alternative geometric interpretation of the conformal group. The conformal group in $\mathbb{R}^{d,1}$ is isomorphic to $SO(d+1,2)$, the Lorentz group of a space with two extra dimensions. Suppose this space $\mathbb{R}^{d+1,2}$ has metric

$$\eta_{ab} = \text{diag}(- + \dots + -)_{ab} \quad (8)$$

where the last two dimensions are the ‘extra’ ones. A light ray in this space can be parameterized by $d + 1$ dimensional coordinates x^μ in the following way:

$$\zeta^a = \kappa(x^\mu, \frac{1}{2}(1 - x^2), \frac{1}{2}(1 + x^2)) \quad (9)$$

where κ is some arbitrary constant. The group $SO(d + 1, 2)$ moves these light rays around. We can interpret these transformations as maps on x^μ , and in fact these transformations are precisely the conformal transformations, as you can check on your own.

Invariants in $\mathbb{R}^{d+1,2}$ should also be conformal invariants, for example

$$\zeta_1 \cdot \zeta_2 = \eta_{ab} \zeta_1^a \zeta_2^b = \frac{1}{2} \kappa_1 \kappa_2 (x_1 - x_2)^2. \quad (10)$$

This statement is not quite true because ζ^a and $\lambda \zeta^a$ are identified with the same x^μ , so κ is a redundant variable. Conformal invariants actually are cross ratios of invariants in $\mathbb{R}^{d+1,2}$, for example

$$\frac{\zeta_1 \cdot \zeta_2 \zeta_3 \cdot \zeta_4}{\zeta_1 \cdot \zeta_3 \zeta_2 \cdot \zeta_4}. \quad (11)$$

This perspective was lifted from a paper by Callan. It may be useful if you are in the business of relating CFTs to theories in higher dimensional spacetimes, but you are otherwise free to ignore it.

2 Constraints on correlators in a CFT

We will now attempt to say something concrete about constraints on a QFT from conformal invariance, namely constraints on correlators. Consider a Green’s function

$$G = \langle \phi_1(x_1) \dots \phi_n(x_n) \rangle \quad (12)$$

where ϕ_i is a conformal primary. Recall from last lecture that this means ϕ satisfies $[\hat{D}, \phi(0)] = -i\Delta\phi(0)$, where Δ is the weight and \hat{D} is the dilatation operator defined above.

How does $\phi(x)$ transform under \hat{D} ? Writing the field as $\phi(x) = e^{-i\hat{P}\cdot x} \phi(0) e^{+i\hat{P}\cdot x}$, and using the following result

$$e^{-i\hat{P}\cdot x} \hat{D} e^{+i\hat{P}\cdot x} = \hat{D} + x \cdot \hat{P} \quad (13)$$

which follows from the algebra, then $[D, \phi(x)] = -i(\Delta + x \cdot \partial)\phi(x)$.

The Green’s function thus transforms under the action of \hat{D} by:

$$\delta G = \sum_{i=1}^n (x_i^\mu \frac{\partial}{\partial x_i^\mu} + \Delta_i) \langle \prod \phi_i(x_i) \rangle + \langle 0 | \hat{D} \prod \phi_i(x_i) | 0 \rangle + \langle 0 | \prod \phi_i(x_i) \hat{D} | 0 \rangle. \quad (14)$$

Assuming that the vacuum is conformally invariant, the last two terms above vanish. We will assume this in the rest of the discussion. $\mathcal{N} = 4$ is a good example where there is a conformally invariant vacuum.

Now let's ignore what we just did with infinitesimal transformations and look at the finite transformations of the green's function:

$$G' = \left\langle \prod_{i=1}^n \left| \frac{\partial x'}{\partial x} \right|_{x=x_i}^{\Delta_i/d} \phi_i(x'_i) \right\rangle \equiv \left(\prod_{i=1}^n \Omega_i^{-\Delta_i/2} \right) \left\langle \prod_{i=1}^n \phi_i(x'_i) \right\rangle \quad (15)$$

where

$$\Omega_i = \begin{cases} \lambda^{-2} & \text{scale transformation} \\ (1 + 2b \cdot x_i + b^2 x_i^2)^2 & \text{special conformal} \end{cases}$$

Conformal invariance tells us exactly how the Green's function must transform. Therefore, up to the Ω_i factors, the Green's function can only depend on conformal invariants I made from x_i^μ . What are the conformal invariants? Assuming ϕ 's are scalars, let's constrain the conformal invariants by using the symmetries of the theory:

- Translation invariance implies I depends only on differences: $x_i - x_j$. Therefore there are $d(n-1)$ numbers I can depend on.
- Rotational invariance implies I depends only on magnitudes of differences: $r_{ij} = |x_i - x_j|$. There are $n(n-1)/2$ of these numbers.
- Scale invariance implies I depends only on ratios of magnitudes r_{ij}/r_{kl}
- Finally special conformal transformations transform r_{ij} by

$$(r'_{12})^2 = \frac{r_{12}^2}{(1 + 2b \cdot x_1 + b^2 x_1^2)(1 + 2b \cdot x_2 + b^2 x_2^2)} = \frac{r_{12}^2}{\Omega_1^{1/2} \Omega_2^{1/2}}. \quad (16)$$

This result implies that only *cross ratios* are invariant: $\frac{r_{ij} r_{kl}}{r_{ik} r_{jl}}$.

There are $n(n-3)/2$ of these cross ratios, so only $n(n-3)/2$ conformal invariants.¹ Note that this implies there are no conformal invariants for $n < 4$, namely the 2-pt and 3-pt function. This is obvious from the fact that you cannot form cross ratios with only 2 or 3 spatial vectors.

2.1 Two-point function

In fact let's look at the 2-pt function for scalar primaries. Its spatial dependence must be completely determined by the logic above:

$$\langle \phi_1(x_1) \phi_2(x_2) \rangle = c_{12} f(|r_{12}|) = c_{12} \frac{1}{r_{12}^{\Delta_1 + \Delta_2}} \quad (17)$$

¹Ginsparg proves why there are $n(n-3)/2$ if you are curious.

where the first equality follows from translation plus rotation invariance, and the second equality follows from scale transformations on both sides. Finally let's apply a special conformal transformation to both sides:

$$\begin{aligned}\Omega_1^{-\Delta_1/2}\Omega_2^{-\Delta_2/2}\langle\phi_1(x'_1)\phi_2(x'_2)\rangle &= c_{12}\left(\frac{1}{r'_{12}\Omega_1^{1/4}\Omega_2^{1/4}}\right)^{\Delta_1+\Delta_2} \\ &= c_{12}\frac{1}{(r')_{12}^{\Delta_1+\Delta_2}}\Omega_1^{-(\Delta_1+\Delta_2)/4}\Omega_2^{-(\Delta_1+\Delta_2)/4}\end{aligned}\quad (18)$$

However using Eq.(17), we already know what $\langle\phi_1(x'_1)\phi_2(x'_2)\rangle$ is. From this we conclude either

- $\Delta_1 = \Delta_2$, OR $c_{12} = 0$

The special conformal transformation was necessary for this result.

As an aside, there is an alternate shorter argument by Cardy which gives the same result without using the annoying formula for the special conformal transformation, though we still have to 'do' one. First we note that the two-point function $\langle\phi_1(x_1)\phi_2(x_2)\rangle$ depends only on $|x_1 - x_2|$, so we can perform a 180 degree rotation to obtain $\langle\phi_2(x_1)\phi_1(x_2)\rangle$, which must give the same answer. Performing a conformal transformation to the two Green's functions above gives

$$\Omega_1^{-\Delta_1/2}\Omega_2^{-\Delta_2/2}\langle\phi_1(x_1)\phi_2(x_2)\rangle = \Omega_2^{-\Delta_1/2}\Omega_1^{-\Delta_2/2}\langle\phi_2(x_1)\phi_1(x_2)\rangle\quad (19)$$

So either $G_{12} = 0$ or $\Delta_1 = \Delta_2$.

Some final comments on the two point function: for a set of fields ϕ_i with $\Delta_i = \Delta_j = \Delta$

$$\langle\phi_i(x_1)\phi_j(x_2)\rangle = \frac{d_{ij}}{r_{12}^{2\Delta}}\quad (20)$$

where d_{ij} must be symmetric by the argument of Cardy. If we assume the operators are hermitian then d_{ij} is also real, and then we can diagonalize d_{ij} . In a unitary theory, the diagonal elements of d_{ij} are norms of states, and positive. By rescaling ϕ_i , we can arrive at a basis where $d_{ij} = \delta_{ij}$.

2.2 Higher point functions

There are no conformal invariants for the three-point function either. Using translation and rotation invariance, we can write the three-point function as

$$G_{123} = \langle\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle = \sum_{a,b,c} \frac{C_{abc}}{r_{12}^a r_{23}^b r_{31}^c}.\quad (21)$$

Scale invariance then implies all terms on the RHS transform in the same way as the LHS, so $a + b + c = \Delta_1 + \Delta_2 + \Delta_3$. You can then check that special conformal transformations give the additional constraint that

$$\begin{aligned}a &= +\Delta_1 + \Delta_2 - \Delta_3 \\ b &= -\Delta_1 + \Delta_2 + \Delta_3 \\ c &= +\Delta_1 - \Delta_2 + \Delta_3\end{aligned}\quad (22)$$

or else $C_{abc} = 0$. The permutation symmetry of r_{ij} also implies C_{abc} is symmetric. We can't say anything more.

For higher point functions, we can say even less, since there exist conformal invariants formed by the cross ratios, and any correlator contains an arbitrary function of the conformal invariants which cannot be determined by the conformal symmetry. For example, for $n = 4$, there are $4 \times 1/2 = 2$ invariants:

$$G(x_1, x_2, x_3, x_4) = \mathcal{F} \left(\frac{r_{12}r_{34}}{r_{13}r_{24}}, \frac{r_{12}r_{34}}{r_{23}r_{14}} \right) \prod_{i < j} \frac{1}{r_{ij}^{\Delta_i - \Delta_j - \Delta/3}} \quad (23)$$

where $\Delta = \sum \Delta_i$ and \mathcal{F} is undetermined function of the two invariants.

The results in this section are true for $D > 1$, and actually for $D = 2$ there are extra symmetries which allow us to learn something about these undetermined functions of invariants. They're often hypergeometric. Also, in principle, there exist similar expressions for non-scalar fields, though John McGreevy has never caught a glimpse of one. He has glimpsed some spectacular things about Wilson loops in this context, though.

3 Conformal Compactifications

Let's take a short detour to conformal compactifications. We'll find that understanding this will be quite useful for understanding the state-operator correspondence, and later for understanding the geometry of AdS. The general goal of such coordinate transformations is to map infinite coordinates to finite ones, thus allowing us to understand the structure of the boundary of spacetime at ∞ . In particular we'd like to find coordinates in which the metric is $\Omega(x)g_{\mu\nu}$ where the coordinates have a finite range because we can take care of the $\Omega(x)$ with a Weyl transformation. Furthermore the presence of the factor of $\Omega(x)$ does not affect the casual structure of spacetime- spacelike vectors remain spacelike, and the same goes for lightlike and timelike.² All this talk makes Penrose diagrams sound way more informative than they may seem.

Start with the simple example of Euclidean space \mathbb{R}^{p+1} with metric $ds^2 = dr^2 + r^2 d\Omega_p^2$ where $d\Omega_p^2$ is the metric on the p -sphere. Define $r = \tan \theta/2$. Then the metric is

$$ds^2(d\theta^2 + \sin^2 \theta d\Omega_p^2) \frac{1}{4 \cos^2 \theta/2} \quad (24)$$

which is the metric on the $p + 1$ -sphere with an extra conformal factor $\frac{1}{4 \cos^2 \theta/2}$. This extra factor blows up at $\theta = \pi$ or $r \rightarrow \infty$ but we don't care! By making this coordinate change, we turned $r \rightarrow \infty$ to a finite point, thus adding a point to \mathbb{R}^{p+1} to get S^{p+1} . It was a one-point compactification.

Moving onto Minkowski $\mathbb{R}^{p,1}$ with $ds^2 = -dt^2 + dr^2 + r^2 d\Omega_{p-1}^2$, by making the coordinate change $t \pm r = \tan \left(\frac{\tau \pm \theta}{2} \right)$ the metric becomes

$$ds^2 = (-d\tau^2 + d\theta^2 + \sin^2 \theta d\Omega_{p-1}^2) \frac{1}{4 \cos^2 \frac{\tau+\theta}{2} \cos^2 \frac{\tau-\theta}{2}} \quad (25)$$

²There may be some trouble at the conformal boundary though?

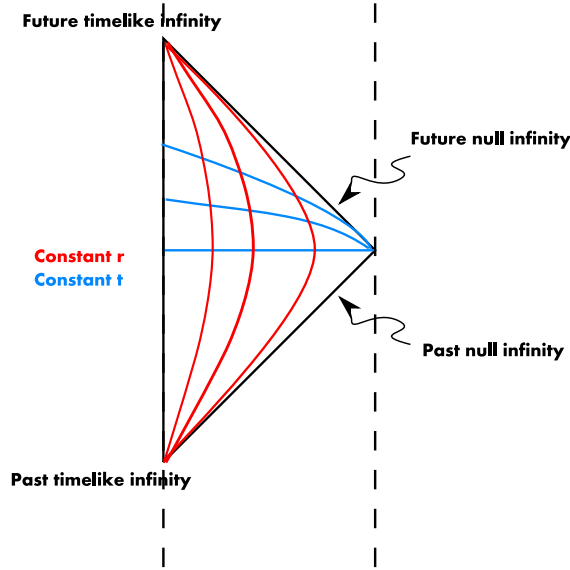


Figure 1: Penrose diagram of Minkowski space, with angular coordinates on S^{p-1} suppressed. The size of the S^{p-1} shrinks at $\sin \theta = 0$. The analytical continuation of the finite τ coordinate is indicated by the dashed lines, and gives the Einstein static universe.

which looks like $\mathbb{R} \times S^p$ with a conformal factor which blows up at the conformal boundary. τ has range $[-\pi, \pi]$ and θ has range $[0, \pi]$. The Penrose diagram is shown in Fig(1).

The reason we went through all this trouble is that a different subgroup of the action of the conformal group is actually obvious in these coordinates. Recall the conformal group of $\mathbb{R}^{p,1}$ is $SO(p+1, 2)$. There are two interesting decompositions (subgroups) of the conformal group:

- $SO(p, 1) \times SO(1, 1)$ where $SO(p, 1)$ is the Lorentz group and $SO(1, 1)$ corresponds to dilations \hat{D} (don't worry about why). If we think about this decomposition in terms of the higher dimension space $\mathbb{R}^{p+1,2}$, the two subgroups act on different directions:

$$\underbrace{(- + \dots + + -)}_{SO(p, 1) \times SO(1, 1)} \quad (26)$$

The Hamiltonian is just the usual generator of time translations and has a continuous spectrum because the field theory lives on \mathbb{R}^p , and we've assumed it's a CFT.

- $SO(p+1) \times SO(2)$ is obvious in the conformal coordinates we discussed above. Here $SO(p+1)$ rotates S^p and $SO(2)$ corresponds to τ translations. In $\mathbb{R}^{p+1,2}$, $SO(p+1)$ rotates the spatial coordinates and $SO(2)$ rotates the time-like coordinates.

$$\underbrace{SO(2)}_{\underbrace{(- + \dots + -)}_{SO(p+1)}} \quad (27)$$

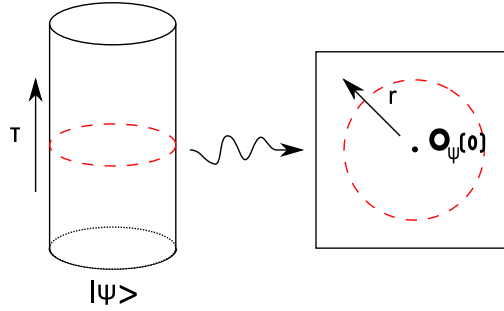


Figure 2: The state operator correspondence. The cylinder on the left represents S^p . Start with an initial state $|\psi\rangle$ and evolve in τ with H_τ . The picture on the right is the space in r coordinates where the red dashed lines are constant τ slices which are also constant r slices.

The τ -translations are generated by a different operator than the one above. Let's call it H_{global} since τ is called the global time. In fact, it's actually equal to

$$H_{global} = \frac{1}{2}(C_0 + P_0) = J_{0,p+2} \quad (28)$$

where C_0 is 0 component of special conformal generator, and $J_{0,p+2}$ is the generator in the space $\mathbb{R}^{p+1,2}$. This time spatial sections of this theory are S^p which implies H_{global} has a discrete spectrum!

One lesson from this is that it's difficult in a CFT to agree on what the Hamiltonian is, because we have a bunch of charges to pick from. The physics looks different in different coordinates though they are of course related by conformal transformations.

4 State-Operator Correspondence

Consider a QFT on $\mathbb{R} \times S^p$, which means time $\tau \times$ a compact space to avoid some IR issues, and so there is a ground state separated by a gap from the first excited state. The metric is $ds^2 = d\tau^2 + d\Omega_p^2$. Make a (large) change of coordinates $r = e^\tau$, then the metric is $ds^2 = (dr^2 + r^2 d\Omega_p^2)/r^2$ which is flat space with a conformal factor.³

The statement of the state-operator correspondence is that given an initial state on the cylinder (S^p) $|\psi\rangle$, we can make a conformal transformation that squashes it all to the origin on the plane (\mathbb{R}^{p+1}). Thus it is a local perturbation on the plane, and there is a corresponding local operator $\mathcal{O}_\psi(0)$ at the origin. We can also go from local operators on the plane back to initial states on the cylinder:

$$|\mathcal{O}\rangle = \lim_{x \rightarrow 0} \mathcal{O}(x)|0\rangle \quad (29)$$

Therefore in a CFT there is a correspondence between local operators and states. See Fig(2).

³OK, we cheated for the sign of $d\tau^2$ in ds^2 but it just means using a slightly different coordinate transformation.