8.821 F2008 Lecture 07

Preview of AdS–CFT Correspondence; String Theory from Gauge Fields; and 't Hooft Counting

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References:

- Maldacena, hep-th/0309246, §2, 3.0;
- MAGOO, §1.2

1 Preview of AdS–CFT Correspondence

Ideology:

$$\begin{array}{rcl} \text{fields in AdS} & \longleftrightarrow & \text{local operators of CFT} \\ \text{spin} & & \text{spin} \\ \text{mass} & & \text{scaling dimension } \Delta \end{array}$$

In particular, for scalars, $m^2 L_{AdS}^2 = \Delta(\Delta - 4)$ [recall that $\lambda = L_{AdS}^4/\alpha'^2$][1]

Last time we discussed the Kaluza-Klein (KK) harmonics of 10d supergravity ('SUGRA') fields (= *massless* string modes). These correspond to superconformal primary operators of the $\mathcal{N} = 4$ theory according to:

$$\Phi(x, y) = \sum_\ell \phi_\ell(x) Y_\ell(y)$$
$$Y_\ell = T_{i_1,\ldots,i_\ell} y^{i_1} \cdots y^{i_\ell} \longleftrightarrow T_{i_1,\ldots,i_\ell} \mathrm{Tr}\,(X^{\{i_1} \cdots X^{i_\ell\}})$$

In which case the mass/scaling dimension correspondence reads:

$$m(\phi_\ell) = \ell/L \longleftrightarrow \Delta \sim \ell$$

[1]We'll show this soon.

On the left hand side, this is just the mass of a KK harmonic on a space of size L, *i.e.* an eigenvalue of the laplacian on the 5-sphere. On the right hand side, we have an operator made from a product of l scalar fields. A 4d scalar has engineering dimension 1, so the dimension of the product is l in the free theory, at least. (It is actually true for all λ by the BPS property.)

The above can be considered as the "highest weight" states. Then, using the fact that the supersymmetry algebras match on the two sides, we can fill out multiplets on both sides by acting with the generators. In this way, the correspondence of all SUGRA modes in the $\mathcal{N}=4$ theory can be found (an intimidating table can be found in ED'H–DZF, hep-th/0201253, p.50).[2]

Now consider some observables of a QFT (we'll assume Euclidean spacetime for the now):

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\cdots\mathcal{O}_n(x_n)\rangle$$

We can write down a generating functional $Z[J]$:

$$\mathcal{L}(x) \to \mathcal{L}(x) + \sum_A J_A(x)\mathcal{O}_A(x) \equiv \mathcal{L}(x) + \mathcal{L}_J(x)$$
$$Z[J] = \langle e^{-\int \mathcal{L}_J}\rangle_{CFT}$$

where $J_A(x)$ are arbitrary functions (sources) and $\{\mathcal{O}_A(x)\}$ is some basis of local operators.

The n-point function is then given by:

$$\langle \prod_n \mathcal{O}_n(x_n)\rangle = \prod_n \frac{\delta}{\delta J_n(x_n)} \ln Z\Big|_{J=0}$$

Since $\mathcal{L}_J$ is a *UV* perturbation ($\because$ it is a perturbation on *bare* Lagrangian by *local* operators), in AdS it corresponds to perturbation near the boundary ($ds^2 = dx^2/r^2 + dr^2/r^2$, boundary at $r=0$). (Recall from the counting of degree of freedom that QFT with UV cutoff $E < 1/\delta \longleftrightarrow$ AdS cutoff $r > \delta$.) The perturbation J will be encoded in the boundary condition on bulk fields.

Incidentally, this resolves a huge confusion in GR literature dating back to 1970's, which claims that "Cauchy problem[3] in AdS is not well-posed." What happens is that signals can reach from the surface where initial data is supplied to the boundary in finite time[4], and hence boundary condition is needed for the problem to be well-posed.

The idea (GKPW) for computing $Z[J]$ is then:

$$Z[J] = \langle e^{-\int \mathcal{L}_J}\rangle_{CFT} = \underbrace{Z_{\text{strings}}[\text{b.c. depends on } J]}_{=???} \underset{\substack{g_s\to 0\\ L^2/\alpha'\to\infty}}{\sim} e^{S_{\text{SUGRA}}}\Big|_{\text{EOM, b.c. depend on } J}$$

Note that the limit $g_s \to 0, L^2/\alpha' \to \infty$ in AdS corresponds to the limit $N\to\infty, \lambda\to\infty$ in QFT.

[2]Note that Tr $(X^{\{i_1}\cdots X^{i_\ell\}})$ are chiral ($\bar{Q}\mathcal{O}=0$) primaries ($K\mathcal{O}=0$), and hence are short (BPS) multiplets. The correspondences of long multiplets is another story which we'll get to soon.

[3]A Cauchy problem is one in which initial data is specified in some space-like slice, and one is asked to determine the future evolution of the system.

[4]We will study the geometry of AdS in detail next week.

As an example of the application of this formula, one can compute the 3-point function of supergravity states. It turns out that the results agree *exactly* with $\mathcal{N} = 4$ SYM (at leading order in $1/N$). A priori this is a surprising statement, since the field theory calculation is done by perturbation theory at small $\lambda = g_{YM}^2 N$, while the gravity calculation is meant to work at large λ. This happens because all the supergravity states correspond to BPS (= "chiral primary") operators, and hence their 3-point functions actually do not depend on λ. This specialness of the states we've been able to map so far may worry you: how do we describe the rest of the field theory operators using the bulk theory? To approach this question, we take a small detour.

2 Strings from Gauge Fields

There are several long-suspected connection between strings and gauge fields.

2.1 Strings as model of hadrons (late 60's)

For hadron with largest spin J for a given mass,

$$J = \alpha' m^2(J) + const. \qquad \text{“Regge trajectory”}$$

which is precisely the form of the spectrum of the vibration modes of a spinning quantum relativistic string in $\mathbb{R}^D$.

2.2 Flux tubes in QCD

We know that the gauge field in QCD is in the confined phase (as opposed to the Coulomb phase, see Fig. 1(a),(b) for illustration). Consequently, the field lines bunch together and behave like a tensionful flux tube.

More concretely, consider the area law for Wilson loop:

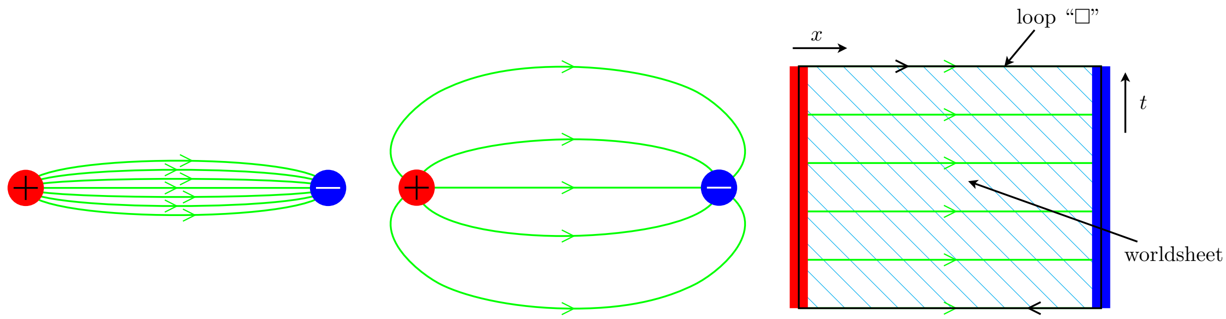
$$\langle \mathrm{Tr}\ \mathcal{P} e^{i \int_\Box A} \rangle \stackrel{?}{=} e^{-\,\mathrm{Area}(\Box)\cdot \mathrm{tension}}$$

Here $\Box$ denotes the worldsheet as shown in Fig. 1(c). This is an order parameter for confinement and indicates that flux tube has a finite tension if and only if the gauge theory is in the confined phase.

The above suggests that we may try to use flux tubes as microscopic variables[5], which should behave as strings in $\mathbb{R}^{3,1}$. However, a result from Polyakov suggests that this is problematic.

Polyakov's result is this: in quantizing the string worldsheet with metric $\gamma_{\alpha\beta}$, the transformation $\gamma_{\alpha\beta} \to e^{\phi(\sigma)}\gamma_{\alpha\beta}$ is a gauge symmetry and hence $\frac{\delta \Gamma_{\text{eff}}}{\delta \phi} = 0$. There is however a conformal anomaly,

[5]It will turn out that a string description *does not* require confinement. The loophole is that one may consider funny shaped worldsheets.



(a) (b) (c)

Figure 1: Flux tubes in QCD: (a) meson in confined phase, (b) ‘meson’ in Coulomb phase, *i.e.* a dipole (c) the world sheet of flux tube for computing Wilson loop.

which generates kinetic term of ϕ in Γ_{eff}. Hence ϕ should be treated as an extra dimension. For QFT in $d \leq 1$, the procedure for quantizing ϕ is known (“Liouville”), and this is really the correct idea (this leads to what is sometimes called ‘old matrix models’, including the $c = 1$ matrix quantum mechanics). In $d = 4$ this result (so far) serves as more inspiration for our extra-dimensional picture..

3 ’t Hooft Counting

The most explicit evidence that gauge theory leads to string theory comes from ’t Hooft counting.

Consider a (any) quantum field theory with *matrix* fields[6] $\Phi^{b=1,\ldots,N}_{a=1,\ldots,\bar{N}}$ (for concreteness we’ll take the matrix group to be $U(N)$), and consider a Lagrangian of the form:

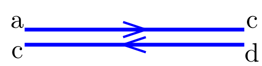
$$\mathcal{L} \sim \frac{1}{g^2_{\text{YM}}} \text{Tr}\, \left((\partial\Phi)^2 + \Phi^2 + \Phi^3 + \Phi^4 + \ldots \right)$$

Rescale $\Phi = g_{\text{YM}}\tilde{\Phi}$,

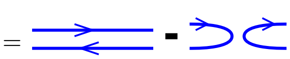
$$\mathcal{L} \sim \text{Tr}\, \left((\partial\tilde{\Phi})^2 + \tilde{\Phi}^2 + g_{\text{YM}}\tilde{\Phi}^3 + g^2_{\text{YM}}\tilde{\Phi}^4 + \ldots \right)$$

Now consider propagators. It is convenient to adopt the *double line* notation, in which oriented index lines follow conserved color flow, so that, for propagator[7]:

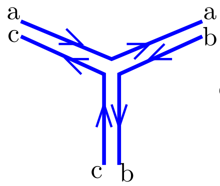
$$\langle \tilde{\Phi}^a_b \tilde{\Phi}^d_c \rangle \propto \delta^a_c \delta^d_b =$$



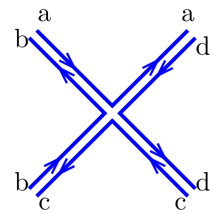
[6]By matrix field we mean that their products appear in the Lagrangian only in the form of matrix multiplication, e.g. $(\Phi^2)^c_a = \Phi^b_a \Phi^c_b$.

[7]Had we been considering $SU(N)$, the result would be $\langle \tilde{\Phi}^a_b \tilde{\Phi}^d_c \rangle \propto \delta^a_c \delta^d_b - \delta^a_b \delta^c_d / N^2 =$ 

And similarly for vertices:



$\propto g_{\rm YM}$



$\propto g_{\rm YM}^2$

Now consider the *'t Hooft limit* in which $N \to \infty$ and $g_{\rm YM} \to 0$ but which $\lambda = g_{\rm YM}^2 N = \text{const}$. Is this limit classical/free? The answer turns out to be no. The loophole is that even though the coupling goes to zero, the number of modes diverges.

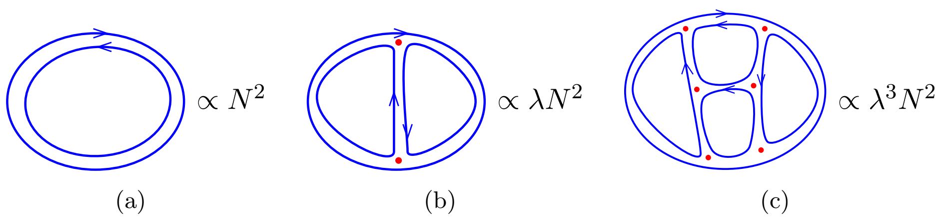
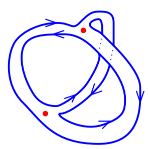


Figure 2: Planer graphs that contribute to the vacuum→vacuum amplitude.



$\propto g_{\rm YM}^2 N = \lambda N^0$

Figure 3: Non-planer graph that contributes to the vacuum→vacuum amplitude.

To see this more concretely, consider vacuum→vacuum diagrams (see Fig. 2 and 3 for illustration). In Fig. 2 we have a set of *planer graphs*, whose contributions take the general form $\lambda^n N^2$. However, there are also contributing non-planer graphs, such as the one in Fig. 3, whose contribution does the take that general form.

Every double-line graph specifics a triangulation of a 2-dimensional surface Σ. There are two ways to construct the explicit mapping:

Method 1 ("direct surface") Fill in index loops with little plaquettes.

Method 2 ("dual surface") (1) draw a vertex in every index loop and (2) draw an edge across every propagator.

These constructions are illustrated in Fig. 4 and 5.

Comparing the resulting surface and the corresponding contribution, it can be seen that the contribution of a graph with h handles is proportional to $N^{2-2h}\lambda^n$. Thus, the partition function (which is also the vacuum→vacuum amplitude) must take the form:

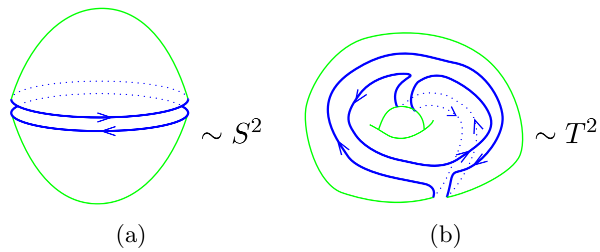


Figure 4: Direct surfaces constructed from the vacuum diagram in (a) Fig. 2a and (b) Fig. 3.

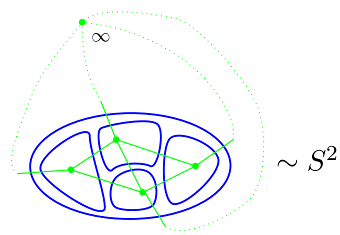


Figure 5: Dual surface constructed from the vaccum diagram in Fig. 2c. Note that points at infinity are identified.

$$\ln Z = \sum_{h=0}^{\infty} N^{2-2h} f_h(\lambda)$$

which is exactly like the result from string perturbation expansion.