8.821 F2008 Lecture 06: Supersymmetric Lagrangians and Basic Checks of AdS/CFT

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We are on our way to talking about really awesome things about supercool stuff. Before we get there, though, we need to develop some very powerful technology. To that end, today we will talk about

1. SUSY Lagrangians and a whirlwind tour of the beauties of superspace.
2. more on $\mathcal{N} = 4$ SYM.
3. Back to the Big Picture: Some basic checks of AdS/CFT

Looking past this lecture, we will be talking about strings from gauge theory next.

1 $\mathcal{N} = 4$ SYM and Other Supersymmetric Lagrangians

Recall that the field content of $\mathcal{N} = 4$ SYM is a vector A_μ, gaugini $\lambda^{I=1...4}$, and six scalars X^i, all in the adjoint of the gauge group. The Lagrangian density (which is completely determined by the amount of SUSY, up to two parameters, (g_{YM}, ϑ)), is

$$\begin{aligned}
\mathcal{L} &= \frac{1}{g_{YM}^2}\left(tr\left[F^2 + (DX^i)^2 + i\bar{\lambda}\not{D}\lambda\right.\right. \\
&- \left.\left.\sum_{i<j}^{6}[X^i, X^j]^2 - \lambda[X,\lambda] + \bar{\lambda}[X,\bar{\lambda}]\ \right]\right) \\
&+ \frac{i\vartheta}{8\pi^2} tr(F \wedge F) \qquad (1)
\end{aligned}$$

The non-zero, $\mathcal{N} = 4$ SUSY transformations are (very schematically!)

$$
\begin{aligned}
[Q, X] &= \lambda \\
\{Q, \lambda\} &= F^{+} + [X, X] \\
\{Q, \bar{\lambda}\} &= DX \\
[Q, A] &= \lambda
\end{aligned} \tag{2}
$$

A Note: There are obviously indices and gamma/sigma matrices suppressed ALL over the place (Lorentz vector, Lorentz spinor and $SO(6)$ vector/spinor, supersymmetry). If you want to put the indices in, either leave that as a fun exercise, or check out Weinberg, volume 3. As an example, $F^{+} \equiv \sigma^{\mu\nu} F_{\mu\nu}$.

1.1 A Superspace Detour

The $\mathcal{N} = 4$ SYM Lagrangian is an example of a (highly) supersymmetric Lagrangian. So far, I just told you what it was and that it was SUSY invariant (something you could sit down in the privacy of your office and check, if you wanted). It'd be nice, though, if there were some sort of a machine that one could crank to generate supersymmetric lagrangians. That crankable machine is superspace.

To understand why superspace is useful, we should think about why fields are useful for representing translationally invariant Lagrangians in ordinary QFT. One reason is that the representations of the translation group on the fields are particularly simple

$$
\phi(x) = e^{i\hat{P}\cdot x}\phi(0) \tag{3}
$$

We'd like to introduce a superfield (that comes with its own supercapitalization) $\Phi(x, \theta)$, which is now a function of spacecoordinates x and "superspace" coordinates θ, that well represents translational invariance AND supersymmetry

$$
\Phi(x, \theta) = e^{i\hat{P}\cdot x + i\hat{Q}\cdot\theta}\Phi(0, 0) \tag{4}
$$

where the $\hat{Q}'s$ are the operators that generate supersymmetry transformations.

Now, in QFT, one can automatically get translationally invariant actions

$$
S = \int d^d x\, \mathcal{L}(\phi, \partial\phi) \tag{5}
$$

as long as $\partial_x \mathcal{L} = 0$. Similarly the (here unproven) claim is that

$$
\int d^d x\, d^{\mathcal{N}\cdot s}\theta\, \mathcal{L}(\Phi(x, \theta)) \tag{6}
$$

is supersymmetric as long as $\partial_\theta \mathcal{L} = 0$. Here s denotes the smallest (real) dimension of the spinor representation in d dimensions and $\mathcal{N}$, is, as usual the number of supersymmetries. Then $\mathcal{N} \cdot s$ is just the number of real supercharges. For example for $\mathcal{N} = 1$, $d = 4$, $\mathcal{N} \cdot s = 4$, because either a Weyl or Majorana spinor (the minimum in four dimensions) has four real components.

1.1.1 BPS or "Chiral" Multiplets

In lecture 5 we made a big deal about special representations of supersymmetry which are killed by some of the supercharges. Such multiplets have correspondingly special properties in superspace. Consider a field which satisfies

$$\begin{aligned} [\bar{Q}, \Phi] &= 0 \\ [Q, \Phi] &\neq 0 \end{aligned} \tag{7}$$

These multiplets, which are BPS (half of the supersymmetries annihilate them) are generally called chiral multiplets. Sometimes they are actually chiral (in the sense of the Lorentz group), but often times they are not. This follows a long tradition in physics of calling things other things which they are not.

These multiplets are functions of only half of superspace as

$$\Phi(x, \theta, \bar{\theta}) = e^{i(Q\theta + \bar{Q}\bar{\theta})} \Phi(x, 0, 0) = \Phi(x, \theta, 0) \tag{8}$$

Ok, well this equation is not exactly correct (as Senthil pointed out; really the RHS should be $\Phi(y \equiv x + i\bar{\theta}\theta, \theta))$, but it is morally correct: the Φ's are functions of half of superspace. Because of this, it is possible to add terms to the Lagrangian density that are integrated over only half of superspace and maintain supersymmetry:

$$\Delta \mathcal{L} = \int d^2\theta \, W(\Phi) + \int d^2\bar{\theta} \, \bar{W}(\bar{\Phi}) \tag{9}$$

Here, we are explicitly working in $d = 4$, $\mathcal{N} = 1$ superspace. These terms are supersymmetric, as long as W is a holomorphic function of Φ, $\partial_{\bar{\Phi}} W = 0$. With this constraint, W is a function we are free to choose, and is known as the superpotential.

Two examples of a superpotential are

- The second line of equation (1). Here, we refer to the fact that this line can be written in $d = 4$, $\mathcal{N} = 1$ superspace (where we pick out a particular $\mathcal{N} = 1$ subgroup from the $\mathcal{N}$=4). In a certain sense, one of the λ's and F can be thought of as comprising one $\mathcal{N} = 1$ chiral multiplet, while the remaining three λ's and six λ's can be thought of as another three chiral multiplets. The second line of equation (1) is a superpotential for these three chiral multiplets. On pset 2 you will have a chance to think about this more precisely.

- The gauge kinetic terms and third line of equation (1) can be thought of as coming from the $\mathcal{N} = 1$ superpotential

$$\int d^2\theta \, \tau \, tr(\lambda_\alpha \lambda^\alpha) \tag{10}$$

where the λ appearing is the superpartner of F (α is a spinor index), and τ is a complexified coupling constant $\tau \equiv \frac{4\pi i}{g_{YM}^2} + \frac{\vartheta}{2\pi}$. Here λ should be thought of as a superfield whose lowest component is the gaugino. The superfield expansion contains a $\lambda = \ldots \theta F$ term. Multiplying two together, one gets

$$\frac{1}{g_{YM}^2} F^2 \subset \int d^2\theta \, \tau \, \theta^2 F^2 \subset \int d^2\theta \, \tau \, tr(\lambda_\alpha \lambda^\alpha) \tag{11}$$

The rest of the gauge kinetic terms and theta angle term can be understood similarly.

1.2 Holomorphy and Non-Renormalization (aka Seibergology)

What's the big deal with this newfangled "superpotential"?

Well, as we said before, the superpotential has to be holomorphic in order for supersymmetry to be preserved. We can take this line of reasoning one step further–we can think of promoting the couplings to dynamical superfields (whose lowest component vevs are just the constant couplings). Then, the superpotential must be holomorphic, also, in the couplings.

This statement is uncomfortably powerful. For example, it implies that if SUSY is not broken, the form of radiatively generated corrections to the superpotential are severely constrained–they must be holmorphic in the fields and the couplings. For example, one could never generate a term in the superpotential that was a function of both τ and $\bar{\tau}$, $W \neq W(\tau\bar{\tau})$.

This leads to many non-renormalization theorems in supersymmetric field theories, one example of which is for the β function of supersymmetric gauge theories. This says that

$$\beta_{YM} = 1 \text{ loop} + \text{non-perturbative} \tag{12}$$

This makes sense because we know the theta angle cannot appear in the beta function perturbatively. However, since the effective gauge coupling function must be holomorphic in τ, the only perturbative contribution to the beta function can be $O(\tau^0)$, i.e., the one loop contribution.

For $\mathcal{N} = 4$ SYM, one can go home and calculate (for pset 2) that the one loop contribution to the beta function is identically zero. In this theory, the non-perturbative terms can be understood as coming from instantons. However, as it turns out, they do not correct the beta function, but do add higher derivative terms elsewhere in the action[1]. Hence, in this theory the beta function is zero, even nonperturbatively, and the theory really is scale invariant.

[1] This corrects a wrong statement from lecture 4.

For more on Seibergology, see the excellent notes of Argyres:
`http://www.physics.uc.edu/∼argyres/661/index.html`

2 Some More Comments on $\mathcal{N} = 4$ SYM

1. It has a "Coulomb Branch of Vacua":

   The scalar potential in the $\mathcal{N} = 4$ action (1) is:

$$V \propto \sum_{i,j} tr[X^i, X^j]^2 \tag{13}$$

   In general, supersymmetry is unbroken iff $V = 0$, and so there is a "moduli space" of supersymmetric vacua, labeled by vevs of X which satisfy

$$\mathcal{M} = \{X|[X^i, X^j] = 0, \forall i, j\}/\{\text{gauge symmetry}\} \tag{14}$$

   We divide out by the gauge symmetry, since vacua related by gauge rotation are physically equivalent. We can fix this gauge symmetry and satisfy the moduli space condition by picking a basis such that

$$\langle X^{i=1,\ldots,6} \rangle = \text{diag}(x_1^i, \ldots x_N^i) \tag{15}$$

   where N is the number of colors. Hence, the moduli space is just an $6 \cdot N$ dimensional space given by these x's.

   At a generic point in this moduli space $x_m^i \neq x_n^i$, and so the gauge bosons that commute with each $\langle X^i \rangle$ are all just proportional to the $\langle X^i \rangle$'s themselves, and so, at a generic point, the gauge group is broken from $U(N)$ to the $U(1)^N$ subgroup generated by these. Since we have broken to a whole bunch of copies of electromagnetism (with some charged scalars and spinors), this moduli space is known as a Coulomb branch.

   Since the vevs of the X's give a mass scale, not only does a generic point in this moduli space break the gauge symmetry, but it also spontaneously breaks the dilation, and hence, the superconformal symmetry. The higgsing from $U(N)$ to $U(1)^N$ gives W-bosons with a mass derived from this mass scale

$$m_W^{ab} = |x_a - x_b| = Z^{ab} \tag{16}$$

   The W bosons are BPS objects with respect to the still-unbroken $\mathcal{N} = 4$ supersymmetry. This must be true since we said in lecture 5 that the $\mathcal{N} = 4$ BPS massive multiplet (what we have created by higgsing) has the same number of degrees of freedom as the $\mathcal{N} = 4$ massless multiplet (what we had before higgsing). The massless $\mathcal{N} = 4$ gauge multiplet carries its food around with it. The Z above are just the central charges of these BPS objects.

   Recall also that this story has a D-brane interpretation: $\mathcal{N} = 4$ SYM with gauge group $U(N)$ is the worldvolume theory of N (coincident) $D3$ branes. We can think about separating these parallel N $D3$ branes. Since they are parallel, and sit in a ten dimensional spacetime, there are six coordinates which label their positions: x_a^i where i runs over the six transverse dimensions and a runs over N, the number of D branes. Actually, to make sure the mass dimensions of x_a^i, match with those above, we write the separation between brane a and b as

$$\Delta y_{ab}^i = \alpha' |x_a^i - x_b^i| \tag{17}$$

Recall, however, that when the D branes were coincident, we could interpret the lowest (massless) string states which started on one D brane and ended on another as massless gauge bosons for the $U(N)$ gauge theory. As we separate the branes, it is no longer the case that strings stretched between branes are massless: they are stretched. Therefore, the gauge bosons are acquiring a mass! In this context, the Higgs mechanism is just pulling a stack of coincident branes apart–which is kind of superawesome. One can calculate the mass of these W bosons, and not surprisingly they match with the picture above

$$m_W^{ab} = \text{(string tension) x (length)} = \frac{1}{\alpha'}\Delta y_{ab} = |x_a - x_b| \tag{18}$$

The BPS property means that this relation must be true for any value of the string coupling.

2. S-duality (a.k.a. Montonen-Olive)

S-duality says the following absurd thing

$$[\mathcal{N} = 4 \text{ with gauge group } G, \text{coupling } \tau] = [\mathcal{N} = 4 \text{ with gauge group } {}^LG, \text{coupling } -\frac{1}{\tau}] \tag{19}$$

where ${}^LG$ is the "dual" gauge group, more on that in a second. As with many such dualities, it says that these two theories are completely equal, though it might be hard to figure out how to map the observables of one onto observables of the other. The dual group is defined such that the weight lattice of ${}^LG$ is equal to the dual of the weight lattice of G

$$\Gamma_w({}^LG) = (\Gamma_w(G))^* \tag{20}$$

For example,

$$\begin{aligned} {}^LSU(N) &= SU(N)/\mathbb{Z}_N \\ {}^LSO(2N) &= Sp(N) \\ {}^LU(N) &= U(N) \end{aligned} \tag{21}$$

The 'L' is for Langlands.

Really, S-duality is a subset of a larger duality that the theory enjoys: it is invariant under shifts of the theta angle, $\vartheta \to \vartheta + 2\pi$. These two transformations are usually called S and T:

$$S\ (\vartheta = 0) : g_{YM}^2 \to \frac{16\pi^2}{g_{YM}^2} \tag{22}$$

$$T : \vartheta \to \vartheta + 2\pi \tag{23}$$

which together generate the group $SL(2,\mathbb{Z})$:

$$\tau \to \frac{a\tau + b}{c\tau + d}, \quad ad - bc = 1 \tag{24}$$

There are many checks of this duality. For example, on the Coulomb branch, the theory has BPS solitons whose spectrum matches that of the (perturbative) vector multiplet of the dual group. Under an S transformation, the BPS solitons and vector multiplets switch.

3. Superconformal Symmetry

   Supersymmetry and conformal symmetry combine (like Voltron) to form a larger spacetime symmetry group, superconformal symmetry. In particular, the commutator of K (which generates special conformal transformations) and Q (which generates supersymmetry transformations) is nonzero (indices omitted)

$$[K, Q] = S \tag{25}$$

   where S is a fermionic symmetry generator. So, although supersymmetry gave us 16 fermionic symmetry generators, superconformal symmetry enlarges that to 32 fermionic symmetry generators.

3 Basic Checks of AdS/CFT

1. Symmetries

   The two theories (IIB on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ SYM) enjoy all of the same symmetries (as they should if they are the same theory!). For example

   - $SO(4,2) \times SO(6)$: We constructed the AdS_5 metric such that it was $SO(3,1)$ invariant, and also invariant under dilations. We will later see that this generates a whole $SO(4,2)$ symmetry. The $SO(6)$ symmetry is just rotations of the S^5, which is its isometry group. $SO(4,2)$ is also the (bosonic part) of the spacetime symmetry group for the SYM theory. We have also seen in a previous lecture $SO(3,1)$ + dilations are symmetries of the SYM theory (in particular, the dilation symmetry follows from the vanishing of the β function). We will soon see how this too gets enlarged. The $SO(6)$ symmetry here is the previously discussed R-symmetry.
   - 32 fermionic symmetries: Type IIB string theory (and supergravity) in flat space has the maximum number of supercharges allowed, 32. As it turns out (and as will be shown in the next problem set), the $AdS_5 \times S^5$ background does not break any of these symmetries. Similarly, we have just remarked upon how the SYM theory has 32 fermionic supercharges as part of the superconformal group.
   - $SL(2,\mathbb{Z})$: We have just remarked upon the $\mathcal{N} = 4$ SYM $SL(2,\mathbb{Z})$ duality. Similarly, this strong-weak duality is also enjoyed by type IIB string theory. It acts on $\tau = \frac{i}{g_s} + \frac{C_0}{2\pi}$, where C_0 is the Ramond-Ramond zero form. This makes sense, as the S transformation in SYM acts on g_{YM}^2 whereas the S transformation here acts on g_s– we have seen before that we should identify these two quantities under the AdS/CFT duality.

2. Perturbations

The first claim each linearized supergravity perturbation corresponds to some $\mathcal{N}=4$ gauge invariant operator. For now we focus on local operators, combinations of fields taken at the same spacetime point. Since the operators are gauge invariant, they involve a trace, or product of traces over $SU(N)$ indices. We'll focus on single trace operators (the reasons for which will be clear in a future lecture).

We can organize operator representations of the superconformal algebra by starting with operators that are so called "superconformal primaries." These operators $\mathcal{O}$ are annihilated by both K and S

$$[K,\mathcal{O}]=[S,\mathcal{O}]=0 \tag{26}$$

We can then get other operators in the representation by acting with the operators Q and P. Since, for a superconformal primary $[S,\mathcal{O}]=0$, it cannot be written as $[Q, \text{another operator}]$. Glancing at (2), this suggests that all superconformal primaries are built out of the X's. In fact, since commutators appear on the RHS of (2) only symmetric combinations of the X's will be superconformal primaries. Thus we are looking at operators of the form

$$\mathcal{O}^{i_1\ldots i_l} \equiv Tr(X^{\{i_1}\ldots X^{i_l\}}) \tag{27}$$

On the supergravity side, fields (perturbations) on $AdS_5\times S^5$ can be expanded in spherical harmonics on the S^5. For example, using x as coordinates on AdS_5 and y on the S^5 (with $\sum y^2=1$), any field $\Phi(x,y)$ can be expanded as

$$\Phi(x,y)=\sum_l \phi_l(x)Y^l(y) \tag{28}$$

and the spherical harmonics can be written as

$$Y^l(y)=T_{i_1\ldots i_l}y^{i_1}\ldots y^{i_l}|_{\sum y^2=1} \tag{29}$$

The correspondence says that we should identify the spherical harmonics with superconformal primaries as

$$T_{i_1\ldots i_l}y^{i_1}\ldots y^{i_l} \longleftrightarrow T_{i_1\ldots i_l}Tr(X^{\{i_1}\ldots X^{i_l\}}) \tag{30}$$

in a way that will be made precise in not too future lectures. This is enough to organize the whole spectrum of supergravity perturbations. Since we know which supergravity fields correspond to superconformal primaries, we can get descendant operators by acting with Q and P. Similarly, we can get the supergravity perturbations that correspond to these operators by acting with the corresponding symmetries in $AdS_5\times S^5$.