MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics
String Theory (8.821) – Prof. J. McGreevy – Fall 2007

Solution Set 8

Worldsheet perspective on CY compactification

Due: Monday, December 18, 2007 at 11:00 AM in the box or in my office.

1. **Orbifolds.**

Suppose you have in your possession a 2d CFT with the correct central charge to provide (part of) a critical string background. Suppose further that this CFT has a symmetry group Γ (for simplicity let's assume it's discrete, finite and abelian). For example, say the CFT is a sigma model, and the target space has a discrete isometry, which could be a subgroup of a continuous isometry.

Let's try to make a new CFT by gauging this symmetry; in particular, this will mean projecting onto states invariant under Γ.

(a) By considering the torus amplitude of this theory, show that a modular transformation ($\tau \to -1/\tau$) relates the projection onto Γ-invariant states to a sum over new sectors of states called *twisted sectors*. These are sectors of strings which are only closed modulo the action of Γ:

$$x(\sigma + 2\pi) = \gamma x(\sigma)$$

for some $\gamma \in \Gamma$.

The resulting procedure of projecting onto invariant states and summing over twisted sectors in a modular-invariant way is called the *orbifold* construction.

Projecting onto invariant states in the torus vacuum amplitude is accomplished by inserting into the trace the projection operator

$$P = \sum_{\gamma \in \Gamma} U_\gamma$$

where U_γ is the unitary operator that implements the orbifold action on the string hilbert space. Consider a particular term in this sum: it computes

$$\text{tr } U_\gamma e^{-\tau_2 H + i\tau_1 P} = \sum_{n \in \mathcal{H}_{\text{untw}}} \langle n | U_\gamma e^{-\tau_2 H + i\tau_1 P} | n \rangle,$$

where the sum over states in the second line is over the *untwisted sector* **of closed string states quantized on a circle of fixed (euclidean) worldsheet time σ_2; this sector to be distinguished from the twisted sectors we are about to discover. A modular transformation $\tau \to -\frac{1}{\tau}$ interchanges what we consider the space and time dimensions on the worldsheet. Hence it exchanges the circle of constant time on which we construct the hilbert space, with the euclidean time circle that implements the trace. In the dual channel, then we are inserting the operator U_γ at a point in the spatial circle. This is an instruction to act with the orbifold generator γ when going around the loop of the string – the string only needs to close up to the action of gamma. We see that modular invariance of the torus vacuum amplitude demands that we include such sectors.**

If our worldsheet theory can be described in terms of some fields X, we can restate this more simply. The $\tau \to -\frac{1}{\tau}$ modular transformation relates the sector where

$$X(\sigma_1 + 2\pi, \sigma_2) = X(\sigma_1, \sigma_2), X(\sigma_1, \sigma_2 + 2\pi) = \gamma X(\sigma_1, \sigma_2)$$

to the one where

$$X(\sigma_1 + 2\pi, \sigma_2) = \gamma X(\sigma_1, \sigma_2), X(\sigma_1, \sigma_2 + 2\pi) = X(\sigma_1, \sigma_2).$$

The inclusion of such sectors is very natural if we think of the orbifold group as a (discrete) *gauge* **symmetry – configurations related by the action of the gauge group are simply the same, and so on the configuration space of the orbifolded theory, strings that only close up to the action of Γ must be included.**

(b) Consider the simple example where our initial CFT is four free bosons of infinite radius; group them into complex pairs $z^{i=1,2}$, and consider the group $\Gamma = \mathbb{Z}_N$ generated by

$$\gamma : \begin{pmatrix} z^1 \\ z^2 \end{pmatrix} \to \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix} \begin{pmatrix} z^1 \\ z^2 \end{pmatrix}, \quad \omega^N = 1.$$

Show that putting a type II string on this CFT preserves half the supersymmetry. (This is a model of part of a K3 surface.)

For the case of $N = 2$, compute the massless spectrum. Note that γ does not act freely; the geometric space you get by identifying points under gamma is therefore scary to mathematicians; it is called an A_{N-1} singularity. Strings propagation on the other hand is completely fine.[1]

γ is a rotation matrix that lies in $SU(2) \subset SO(4)$. The action of γ defines an action on spinors; this $SU(2)$ subgroup preserves a spinor of $SO(4) \sim SU(2) \times SU(2)$, since the (Dirac) spinor rep is $(2,1) \oplus (1,2)$.

Once we know that there are 16 supercharges preserved, it is easier to compute the spectrum since it must lie in multiplets of the supersymmetry. Depending on whether we start with type IIA or IIB, we get 6d $\mathcal{N} = (1,1)$ or $(2,0)$ supersymmetry (the two numbers are the numbers of left-handed and right-handed supercharges respectively). Let's focus on the IIA case. In this case, the twisted sector contains a massless $(1,1)$ vectormultiplet.

If you want more details about this, ask me.

(c) [BONUS] Note that the orbifolded theory has a new global discrete symmetry, called the *quantum symmetry* (not my fault), which acts with a phase α^k on the sector twisted by ω^k. This symmetry constrains the interactions of the orbifold theory. What can you say about the theory you get by orbifolding by the quantum symmetry?

You get back the original CFT before orbifolding. The states invariant under the quantum symmetry are exactly the untwisted states. The twisted sectors of the orbifold by the quantum symmetry are the states that weren't invariant under the original projection.

2. **GLSM (Gauged Linear Sigma Model).**

   A powerful technique for studying nonlinear sigma models on CY manifolds can be found by studying massive (meaning not scale-invariant) 2d QFTs which RG flow to them in the IR. In particular, the parameter space of the massive theories will give some coordinates on the moduli space of the CFT in the IR.[2]

   Sigma models with Kähler target have 2d $(2,2)$ supersymmetry. So to find good candidate models, let's look at the multiplets of $(2,2)$. The most accessible are

[1] To be precise, it can be shown that the orbifold CFT has a certain nonzero NSNS B_{mn} in the background which resolves the singularity. By deforming this to zero it is actually possible to make the perturbative string theory singular. In that limit some wrapped D2-branes become light and can enhance the gauge symmetry (the abelian gauge symmetry coming from the RR vectors) to $SU(N)$.

[2] The reference for this discussion is E. Witten, "Phases of $\mathcal{N} = 2$ theories in two dimensions." hep-th/9301042.

are
(i) the chiral multiplet, $\bar{D}_\alpha \Phi = 0$, which can be parametrized as

$$\Phi(y, \theta) = \phi(y) + \theta^\alpha \psi_\alpha + \theta^\alpha \theta_\alpha F$$

where ϕ is complex, $\alpha = \pm$, and F is an auxiliary field.

(ii) the vector multiplet, $V = \bar{V}$, which contains a vector, gaugino and auxiliary field D. It can be described in terms of a *twisted* chiral field $\Sigma \equiv \bar{D}_+ D_- V$, satisfying $\bar{D}_+ \Sigma = 0 = D_- \Sigma$. Its expansion is

$$\Sigma = \sigma + \theta^+ \bar{\lambda}_+ + \bar{\theta}^- \lambda_- + \theta^+ \bar{\theta}^- (D + i v_{01}) + ...$$

where v_{01} is the field strength of the $U(1)$ gauge field.

In two dimensions, even abelian gauge theories confine – the maxwell term is by dimensional analysis a relevant operator. Consider a $U(1)$ gauge symmetry, with $N+1$ chiral multiplets of charge 1. An important term in the lagrangian will be the Fayet-Ilioupoulos (FI) term; this is a term

$$= \int d\bar{\theta}_+ d\theta_- \Sigma t = -rD + \theta v_{01}$$

with $t = r + i\theta$

The whole bosonic potential so far is

$$L_D = \frac{D^2}{2e^2} + D \sum |\phi_i|^2 - Dr.$$

(a) Integrate out the auxiliary field D to find the bosonic potential. Show that when r is large and positive (we will denote this regime by the awkward symbol '$r \gg 0$'), the ϕs can't simultaneously vanish and keep zero energy. Convince yourself that the vacuum manifold is therefore $\mathbb{CP}^N$. It is then reasonable to conjecture that this gauge theory flows to a nonlinear sigma model on $\mathbb{CP}^N$ with a size related to the FI parameter.

The terms in the lagrangian involving the D-term are

$$L_D = \frac{D^2}{2e^2} + D \sum_{i=1}^{N+1} |\phi_i|^2 - Dr.$$

The equation of motion for D is algebraic:

$$0 = \frac{\delta L}{\delta D} \Longrightarrow D = -e^2 \left(\sum_{i=1}^{N+1} |\phi_i|^2 - r \right).$$

Plugging this back into L_D gives

$$L_D| = -\frac{e^2}{2} \left(\sum_{i=1}^{N+1} |\phi_i|^2 - r \right).$$

When $r \gg 0$, zero D-term energy then requires that not all of the ϕs vanish. The vacuum manifold is

$$\mathbb{CP}^N = \{ \sum_{i=1}^{N+1} |\phi_i|^2 = r > 0\}/U(1) = \left(\{\phi^i\} - \{0\} \right) /\mathbb{C}^\star.$$

(b) Show that the beta function for the FI parameter is proportional to the sum of the charges of the chiral fields. Relate this to your expectations about the RG flow of the $\mathbb{CP}^N$ model.

The FI coupling appears in the lagrangian as the coupling constant for a single D field:

$$rD.$$

It appears in other terms related to this one by supersymetry. A term which can generate corrections to this one is

$$q_i |\phi^i|^2 D$$

(where q_i is the $U(1)$ charge of ϕ^i since the contraction of a pair of ϕs contains a c-number piece which will renormalize r. The propagator for ϕ in momentum space is

$$\frac{1}{p^2 + m_\phi^2};$$

the mass of the ϕ field is $q_i^2 \sigma^2$. So the renormalization of r at one loop is

$$\delta r = \sum_i q_i \int^\Lambda \frac{d^2 p}{p^2} \frac{1}{p^2 + m_{\phi^i}^2}$$

Introducing a UV cutoff Λ, this is

$$\delta r \propto \sum_i q_i \ln \frac{\Lambda}{m_\phi^2}.$$

The beta function is then

$$\beta_r \sim \Lambda \frac{d}{d\Lambda} \delta r \propto \sum_i q_i.$$

When the sum of the charges is zero, the FI parameter does not run. We will see below that this is the same as the CY condition on the geometry.

Actually, the renormalization of r can be argued to be one-loop exact, since it's related by supersymmetry to the anomaly in the axial R-symmetry.

(c) To get a CY threefold, take $N = 5$ above, and add one more field P of charge -5. Note that the FI parameter no longer runs. Now we can add a gauge-invariant superpotential

$$W = PG_5(\Phi).$$

Find the bosonic potential. In the case where G is transverse, show that the vacuum manifold is $\{P = 0\}$ and

$$\{G_5(\phi)\} \in \mathbb{CP}^4.$$

The fluctuations perpendicular to this locus are all massive, and the IR limit seems to give a sigma model on the quintic.

The bosonic potential is $U_D + U_F$ with

$$U_D = -\frac{e^2}{2}\left(-5|p|^2 + \sum_{i=1}^{N+1} |\phi_i|^2 - r\right).$$

and

$$U_F = \sum_{chirals, f} |\frac{\partial W}{\partial f}|^2 = |G_5(\phi)|^2 + |p|^2 \sum_i |\frac{\partial W}{\partial \phi^i}|^2.$$

For $r \gg 0$, the D-term potential can only vanish if at least one of the ϕs is nonzero. If G is transverse (meaning $G = dG = 0$ has no

solutions other than $\phi^i = 0, \forall i$) then the second term in the F-term potential forces P to vanish. Then the first term (which much vanish independently of the second since they are both positive) forces the massless modes onto the locus $G_5 = 0$ which is the quintic.

Relate the moduli of the quintic to the marginal parameters of the gauge theory.

Note that the word 'moduli' is a bit too useful in string theory. The word 'moduli' here doesn't refer to the vacuum manifold of the 2d gauge theory (which could be called a 'moduli space' if not for the Coleman-Mermin-Wagner-Hohenberg theorem) . Rather, it refers to deformation parameters determining the geometry of the CY. (These are moduli of the 4d effective theory of strings compactified on the CY.) They correspond (at least some of them) to parameters of the 2d gauge theory that don't run under the RG flow.

In this example, these parameters are the (complexified) FI parameter and the parameters in the superpotential. From part (a) we know that the FI parameter controls the size of the projective space into which we're embedding the quintic, so it controls the kähler class of the CY. The parameters in the superpotential of the 2d gauge theory determine the complex structure moduli of the quintic CY target space of the NLSM to which the gauge theory flows. There are 126 degree-five monomials in 5 variables that could appear in the superpotential; you can get rid of 25 of the coefficients by field redefinitions. This counting is just the same as when we counted complex structure moduli of the quintic hypersurface.

(d) Consider varying the complex structure moduli (via the coefficients of G_5) so that G_5 is not transverse, *i.e.* where $G = dG = 0$ *does* have a solution for ϕ not all zero. What happens to the vacuum manifold in this case?

If G is not transverse, *i.e.* there are values of $\phi^i = \phi^i_\star \neq (0, .., 0)$ which solves $G = dG = 0$, then p is not forced to vanish when $\phi^i = \phi^i_\star$. That is, the vacuum manifold has a *branch structure* and is not a manifold. Even worse, the extra branch whose coordinate is p is not compact, unlike the quintic. The path integral then contains an extra integral over the zero mode of p which will make some correlators diverge. This GLSM can be expected to flow to a singular CFT.

(e) There's another semiclassical limit where $|r|$ is large, namely $r \ll 0$. Convince yourself that the D term condition implies that P must have an expectation value in this limit. Show that the resulting effective theory is a *Landau-Ginzburg orbifold*, a model of massless fields with a superpotential

$$W \propto G_5(\phi)$$

orbifolded by a $\mathbb{Z}_5$ symmetry. This is a completely non-geometric CFT, connected by variation of Kähler moduli to the quintic NLSM.

Now the D-term condition forces p not to vanish. We can solve the D-term for p and use the $U(1)$ to pick the phase of p so that it's positive:

$$\langle p \rangle = +\sqrt{r}.$$

The second term in the F-term potential then forces (for transverse G) all the ϕs to have vanishing vevs. Although the ϕs have vanishing vevs, they do not have mass terms. Such a model where massless fluctuations are governed by a higher-order potential is called a Landau-Ginzburg theory. The remaining superpotential is

$$W = PG_5(\phi) = \sqrt{r}G_5(\phi),$$

where we have set P to its vev which we can do because its fluctuations are massive.

There is just one remaining subtlety. Since P has charge -5, its vev does not completely break the gauge symmetry. Gauge transformations

$$\phi \to \omega\phi, P \to \omega^{-5}P$$

with $\omega^5 = 1$ do not change the vev of P and hence still act on the ϕ fluctuations. This discrete gauge symmetry implements a $\mathbb{Z}_5$ orbifold.

(f) Remember that the vectormultiplet contains a complex scalar σ. Under the relation to 4d $\mathcal{N} = 1$, these modes related to the components of the 4d gauge field along the two dimensions along which we reduced. This tells us that there must be potential terms of the form

$$V \supset |\sigma|^2 \left(\sum_i |\phi_i|^2 + 25|p|^2 \right).$$

When $r \gg 0$, $\sum |\phi_i|^2 \sim r$ means that σ gets a large mass; similarly when $r \ll 0$, $|p|^2 \sim r$ gives σ a large mass. What happens in between? Naively at

$r = 0$, we can set $\phi_i = p = 0$ consistent with $D = 0, dW = 0$, so it would seem that σ becomes a flat direction there! Such an extra branch of the target space of the sigma model at a point in its parameter space would lead the correlators of the CFT to diverge there. [3]

The expectation that there should be a singularity where σ becomes a flat direction is correct. However, since the parameters of the GLSM must describe vevs of fields in a supersymmetric 4d effective theory, we know they must come in complex pairs. The complex partner of the FI parameter is the theta angle $\int \theta v_{01}$, as indicated before part (a) of this problem. When $\theta \neq 0$, it forces there to be a nonzero electric field in the vacuum on the Coulomb branch (the 'Coulomb branch' refers to possible vacau where the ϕs (which are charged under the U(1) and hence Higgs fields) are zero, but σ which is neutral can have a vev). This electric field costs energy and lifts the Coulomb branch. Hence there is just one special value of r, θ where a new noncompact σ branch sticks out of our vacuum manifold.

(g) We saw above in section (d) how to make a GLSM for an orbifold. Consider the GLSM with one $U(1)$ and three chiral fields (Z^1, Z^2, P) of charges $(1, N-1, -N)$. Show that the FI parameter is marginal. Show that when $r \ll 0$, this flows to the orbifold of $\mathbb{C}^2$ studied in problem 1. Interpret the variation of the FI parameter as condensing a mode of the twisted sector (this is called *blowing up the singularity*). In the case $N = 2$, show that in the space at $r \gg 0$, the orbifold point has been replaced by a $\mathbb{P}^1$. It is this 2-sphere that shrinks in making the A_1 singularity, and this is why a D2-brane can become massless there.

The sum of the charges is $1 + N - 1 - N = 0$ so the FI parameter does not run. When $r \ll 0$, the vev of P is forced to be nonzero by the D-term, and breaks the gauge symmetry down to $\mathbb{Z}_N$. This $\mathbb{Z}_N$ acts on $Z^{1,2}$ exactly as in problem 1c). The FI parameter can be interpreted as the vev of a twisted sector field because it acts as a chemical potential for vortices of the gauge field.

(i) [BONUS] Compute the Witten index of the gauge theory with $N + 1$ chiral fields based on our answer for the vacuum manifold at large radius. This means that there are at least $N + 1$ supersymmetric vacua. What happens to them in the IR when the $\mathbb{CP}^N$ shrinks to nothing?

[3] Actually the correlators only diverge if the theta angle is also zero, because otherwise there is a constant electric field in the vacuum.

Recall that in the calculation of the beta function for the FI parameter, the mass of the ϕ fields depended on σ. In the case where $\sum_i q_i \neq 0$, the running of the FI parameter can be thought of as an effective potential for σ. In a 2d $(2,2)$ theory, such a potential comes from a *twisted chiral superpotential*:

$$\int d^2\tilde{\theta}\tilde{W}(\Sigma)$$

where $\int d^2\tilde{\theta} \equiv d\theta_+ d\bar{\theta}_-$ is an integral over a different half of superspace (the half that doesn't kill a twisted chiral superfield $\bar{D}_-\Sigma = 0 = D_+\Sigma$), and $\Sigma \equiv \bar{D}_- D_+ V$ is a twisted chiral superfield made from the vectormultiplet whose lowest component is σ. The twisted chiral superpotential which encodes the running of the FI parameter is

$$\tilde{W}(\Sigma) = t\Sigma - \sum_i q_i \Sigma \ln \Sigma/\Lambda;$$

the first term is just the tree-level FI parameter.

For $\mathcal{P}^N$, $\sum_i q_i = N+1$. The condition on Σ for there to be a supersymmetric vacuum is

$$0 = \frac{\partial \tilde{W}}{\partial \sigma} = t - (N+1)\ln \Sigma/\Lambda + N + 1$$

which is solved by

$$\langle \sigma \rangle \sim \Lambda e^{t/N+1} \omega_{N+1}$$

where ω_{N+1} is an $N+1$th root of unity, of which there are $N+1$. This is where the vacua go – the squirt out onto the sigma branch.