MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics
String Theory (8.821) – Prof. J. McGreevy – Fall 2007

Solution Set 6

Superstrings

Reading: Polchinski, Chapters 10, 11, 12, except for the bits about unoriented strings.

Due: Tuesday, November 20, 2007 at 11:00 AM in lecture or in the box.

1. **Bosonization for interacting theories.**

   There are many ways to deform the free boson = free complex fermion theory, besides the linear dilaton deformation discussed in class.

   (a) Change the radius of the boson. What operator accomplishes this in the boson theory? Describe this operator in terms of the fermions. Adding this operator to the free fermion lagrangian makes it no longer free (though still solvable); it's called the *Thirring model.*

   To change the radius of a boson X, we add to the lagrangian the operator
$$\Delta L = \delta R^2 \partial X \bar{\partial} X,$$
   which has dimension (1,1) and therefore can be added to the action (at least with an infinitesimal coefficient) without spoiling conformal invariance. That it can be added to the action with a finite coefficient is a more nontrivial thing in general; in this case, the theory with any value of R is clearly still free, and still a CFT. Notice that this X is the sum of left-moving and right-moving fields: $X(z,\bar{z}) = X_L(z) + X_R(\bar{z})$. Also, notice that $\partial X = -ij, \bar{\partial} X = -i\tilde{j}$ is the translation current for the boson. This deformation of the action is
$$\Delta L = -\delta R^2 j \tilde{j}.$$

   When the boson arises by bosonizing complex fermions of both chiralities,
$$\psi = e^{iH}, \quad \tilde{\psi} = e^{i\tilde{H}}$$
$$\psi^\star = e^{-iH}, \quad \tilde{\psi}^\star = e^{-i\tilde{H}},$$

and there are again left-moving and right-moving currents:

$$j = \psi^\star \psi, \qquad \tilde{j} = \tilde{\psi}^\star \tilde{\psi},$$

which map to the momentum currents above. In terms of the fermions, the deformation is therefore

$$\Delta L = -\delta R^2 j \tilde{j} = -\delta R^2 \psi^\star \psi \tilde{\psi}^\star \tilde{\psi},$$

a four-fermion interaction.

(b) Add a mass for the fermions. Describe this operator in terms of the bosons. The result is called the *sine-Gordon* model.

If you get stuck here, or want to learn more, see page 246 of Coleman, *Aspects of Symmetry*, or *Phys. Rev.* **D11** (1975) 2088.

The mass term for the fermions is something like

$$\Delta L = -m(\psi \tilde{\psi} + \psi^\star \tilde{\psi}^\star)$$

(where I've chosen to break the vectorlike fermion number $\psi \to \alpha\psi, \tilde{\psi} \to \alpha\tilde{\psi}$ and preserve the axial fermion number $\psi \to \beta\psi, \tilde{\psi} \to \beta^{-1}\tilde{\psi}$). In terms of bosons, this is

$$\Delta L = -m \left(e^{iH} e^{i\tilde{H}} + e^{-iH} e^{-i\tilde{H}} \right)$$

since $H, \tilde{H}$ have no OPE singularity, this is

$$\Delta L = -m \left(e^{i\left(H + \tilde{H}\right)} + e^{-i\left(H+\tilde{H}\right)} \right) = -2m \cos H(z, \bar{z}),$$

a sine-gordon interaction.

A few comments:

(i) If we had chosen the mass term to break the axial current instead, this would have been a condensate of a winding mode,

$$\cos \hat{H}, \qquad \hat{H}(z, \bar{z}) \equiv H(z) - \tilde{H}(\bar{z}).$$

(ii) This operator $\cos H$, unlike the change-of-radius operator is not dimension (1,1), at least in the absence of the four-fermion term. It is a relevant operator which drives an RG flow towards a mass gap.

The fact that the dimension of the cosine depends on the radius of the boson (and in particular can vary from irrelevant to relevant) is the origin of the Kosterlitz-Thouless phase transition of the XY model in two spatial dimensions (see *e.g.* **Chaikin and Lubensky** *Principles of Condensed Matter Physics*, **§9.2)**

2. **Operator algebra of spin fields.**

(a) Using bosonization and constraints from Lorentz invariance, convince yourself that

$$\psi^\mu(z)\Theta_\alpha(0) \sim \frac{1}{\sqrt{2}}\frac{1}{\sqrt{z}}\Gamma^\mu_{\alpha\beta}\Theta_\beta(0)$$

and

$$\Theta_\alpha(z)\Theta_\beta(0) \sim C_{\alpha\beta}\frac{1}{z^{5/4}} + \frac{1}{\sqrt{2}}\frac{1}{z^{3/4}}(C\Gamma^\mu)_{\alpha\beta}\psi_\mu(0)$$

where C is the charge conjugation matrix $C\Gamma^\mu C^{-1} = -(\Gamma^\mu)^T$. It may help to note that these are closely related to eqns (12.4.7) and (12.4.18) of Polchinski, and that the power of z is determined by dimensional analysis. Note that the first term in the second equation vanishes when the Θs have the same chiralities, such as in part (b) of this problem.

If you want to worry about signs and factors, it might be helpful to come to terms with the equations of p. 77 of Peskin's notes, where he gives an explicit prescription for the cocyles for the spin fields.

First, bosonize:

$$\frac{1}{\sqrt{2}}\left(\psi^{2a} \pm i\psi^{2a+1}\right) = e^{\pm iH^a}, \quad \frac{1}{\sqrt{2}}\left(\pm\psi^0 + i\psi^1\right) = e^{\pm iH^0}$$

$$\Theta_\alpha = e^{i\alpha_a H^a}$$

where $\alpha_a = \pm\frac{1}{2}$**.** H **is a free boson so**

$$: e^{i\alpha\cdot H} : (z) : e^{i\beta\cdot H} : (0) = z^{\alpha\cdot\beta}e^{i(\alpha+\beta)\cdot H(0)} + ...$$

where note that I'm using as indices the weight vectors α **(sometimes called** s**).**

The fusion of the spinor and the vector gives

$$e^{\pm iH^a(z)}\Theta_\alpha(0) \sim z^{\alpha_a}e^{i(\alpha_b+e_b)H^b}$$

where e_b is a vector of zeros with a 1 in the bth entry. If $\alpha_a = -\frac{1}{2}$, this is singular, and in that case $\alpha_a + e_a = +\frac{1}{2}$; the spin has been flipped. If $\alpha_a = +\frac{1}{2}$, this OPE is nonsingular. Therefore, the singular part of the OPE is exactly the action of the Γ matrices in the creation-annihilation basis we contructed (see Appx B of Polchinski)

$$\Gamma^{0\pm} = \frac{1}{2}\left(\pm\Gamma^0 + \Gamma^1\right), \quad \Gamma^{a\pm} = \frac{1}{2}\left(\pm\Gamma^{2a} \pm i\Gamma^{2a+1}\right)$$

and we find

$$\psi^\mu(z)\Theta_\alpha(0) \sim \frac{1}{\sqrt{2}}\frac{1}{\sqrt{z}}\Gamma^\mu_{\alpha\beta}\Theta_\beta(0) + \mathcal{O}(\sqrt{z})$$

$$\begin{aligned}\Theta_\alpha(z)\Theta_\beta(0) &= e^{i\alpha_a H^a}(z)e^{i\beta_b H^b}(0).\\ &\sim z^{\alpha_a\beta_a}e^{i(\alpha_a+\beta_a)H^a(0)} + \ldots\end{aligned}$$

Only if $\alpha_a = -\beta_a$ for a given a – *i.e.* if the spins along that plane differ – does a given H_a make a singular contribution to the OPE. The possible values of the exponent $\sum_a \alpha_a\beta_a$ are $-5/4, -3/4, -1/4, 1/4, 3/4, 5/4$ depending on how many components of α and β differ. If we allow the Θs to have different chirality (*i.e.* different parity of the number of up spins), then all five can be different, which gives a term going like

$$z^{5/4}C_{\alpha\beta}$$

– $C_{\alpha\beta}$ is the matrix in the spin space which is only nonzero if all the components of α and of β are different.

If α and β have the same chirality, the most singular bit comes from flipping four out of five spins. The power of z from this is

$$-\frac{1}{4} - \frac{1}{4} - \frac{1}{4} - \frac{1}{4} + \frac{1}{4} = -\frac{3}{4}.$$

This leaves a single component of α and β the same, and that entry (say $\alpha_b = \beta_b$) will add instead of cancelling in $\alpha_a + \beta_a$, this entry will add up to:

$$\alpha_b + \beta_b = 2\beta_b = \pm 1$$

which gives

$$e^{i\sum_a(\alpha_a+\beta_a)H^a} = e^{i2\alpha_b H^b} \quad (\textit{no sum on } b).$$

The object on the RHS here is exactly

$$e^{i2\alpha_b H^b} = \frac{1}{\sqrt{2}}\left(\psi^{2b} \pm i\psi^{2b+1}\right)$$

(where we make an exception when $b = 0$ to allow for the existence of time (whose idea was that?)). Since the Lorentz-covariant object which intertwines between the vector and spinor representations is exactly a gamma matrix, we have

$$\Theta_\alpha(z)\Theta_\beta(0) \sim C_{\alpha\beta}\frac{1}{z^{5/4}} + \frac{1}{\sqrt{2}}\frac{1}{z^{3/4}}(C\Gamma^\mu)_{\alpha\beta}\psi_\mu(0) + \frac{1}{z^{1/4}}\mathcal{C}_{\alpha\beta} + \text{regular}$$

where $\mathcal{C}_{\alpha\beta}$ is a matrix which only has nonzero elements if α and β differ by three spins, and I suppose has two fermions in it – again, the first and third terms are zero if the Θs have the same chirality, which they always will in a sensible CFT, exactly because these terms would prevent a local OPE between Θs of opposite chirality. This is yet another point where we are saved from dragons by the GSO projection.

(b) Verify the algebra of spacetime supersymmetry generators

$$Q_\alpha = \oint \frac{dz}{2\pi i}\mathcal{V}_\alpha(z) = \oint \frac{dz}{2\pi i}e^{-\phi/2}\Theta_\alpha(z).$$

By the usual contour argument, using the mysterious fact that $\mathcal{V}_\alpha\mathcal{V}_\beta = -\mathcal{V}_\beta\mathcal{V}_\alpha$, we have

$$\{Q_\alpha, Q_\beta\} = \oint_{C_0}\frac{dz}{2\pi i}\oint_{C_z}\frac{dw}{2\pi i}\mathcal{V}_\alpha(w)\mathcal{V}_\beta(z)$$

where C_z is a contour centered on z. The $\mathcal{V}\mathcal{V}$ OPE, which is the $\Theta\Theta$ OPE in part a (for Θs of the same chirality) times

$$: e^{-\phi(w)/2} :: e^{-\phi(z)/2} :\sim \frac{1}{(w-z)^{1/4}} : e^{-\phi(z)} : + ...$$

gives

$$\{Q_\alpha, Q_\beta\} = \oint_{C_0}\frac{dz}{2\pi i}\oint_{C_z}\frac{dw}{2\pi i}\frac{1}{\sqrt{2}}\frac{1}{w-z}e^{-\phi(z)}(C\Gamma^\mu)_{\alpha\beta}\psi_\mu(z) + ...$$

$$= \oint_{C_0} \frac{dz}{2\pi i} \frac{1}{\sqrt{2}} e^{-\phi(z)} (C\Gamma^\mu)_{\alpha\beta} \psi_\mu(z).$$

This is the momentum generator in a different picture, *i.e.* **an object which when you act on it with a picture changing operator (which will always be available in an amplitude where you have an opportunity to use this algebra, and the answer is independent of where we put the PCOs) you get $\oint \partial X$. This is true because**

$$P_{+1} e^{-\phi} \psi_\mu(z) = \lim_{z' \to z} e^{\phi} T_F(z') e^{-\phi} \psi_\mu(z) = \lim_{z' \to z} (z' - z) \frac{1}{z' - z} i \sqrt{\frac{2}{\alpha'}} \partial X_\mu(z)$$

The momentum of the (left-moving bit of the) closed string is

$$p^\mu = \frac{1}{\alpha'} \oint \frac{dz}{2\pi} \partial X^\mu$$

which gives

$$P_{+1}\{Q_\alpha, Q_\beta\} = \frac{1}{\sqrt{2}} (C\Gamma^\mu)_{\alpha\beta} \oint_{C_0} \frac{dz}{2\pi i} i \sqrt{\frac{2}{\alpha'}} \partial X^\mu = \frac{1}{\sqrt{\alpha'}} (C\Gamma^\mu)_{\alpha\beta} p_\mu.$$

I don't have anything useful to say about the $\sqrt{\alpha'}$.

3. **A little more superstring scattering.**

   Polchinski, problem 12.8.

   This problem asks us to study the tree-level scattering of three closed superstring bosons, two RR and one NSNS. The amplitude, with pictures chosen to saturate the sphere ghost anomaly of -2 on left and right, is

$$\mathcal{A}^{(3)}_{S^2} = \left\langle c\tilde{c} V_1^{(-\frac{1}{2}, -\frac{1}{2})}(z_1) c\tilde{c} V_2^{(-\frac{1}{2}, -\frac{1}{2})}(z_2) c\tilde{c} V_3^{(-1,-1)}(z_3) \right\rangle_{S^2}$$

   where the vertex operators are

$$V^{(-\frac{1}{2}, -\frac{1}{2})}(z) = g_c e^{-\phi/2 - \tilde{\phi}/2} \Theta_\alpha \tilde{\Theta}_\beta e^{ik\cdot X} F^{\alpha\beta}$$

   and

$$V^{(-\frac{1}{2}, -\frac{1}{2})}(z) = g_c e^{-\phi - \tilde{\phi}} \psi^\mu \tilde{\psi}^\nu e^{ik\cdot X} \zeta_{\mu\nu}.$$

The correlators we need are

$$\langle c\tilde{c}(z_1)c\tilde{c}(z_2)c\tilde{c}(z_3)\rangle_{S^2} = C^{gh}_{S^2}|z_{12}z_{23}z_{31}|^2$$

$$\langle e^{-\phi/2-\tilde{\phi}/2}(z_1)e^{-\phi/2-\tilde{\phi}/2}(z_2)e^{-\phi-\tilde{\phi}}(z_3)\rangle_{S^2} = C^{sgh}_{S^2}|z_{12}|^{-1/2}|z_{13}z_{23}|^{-1}.$$

$$\langle\prod_{i=1}^{3} : e^{ik_i\cdot X} :\rangle_{S^2} = C^X_{S^2}\prod_{i<j}|z_{ij}|^{\alpha' k_i\cdot k_j}\tilde{\delta}^{10}(\sum k).$$

$$\langle\Theta_\alpha(z_1)\Theta_\gamma(z_2)\psi^\mu(z_3)\rangle_{S^2} = z_{12}^{-3/4}z_{13}^{-1/2}z_{23}^{-1/2}\frac{1}{\sqrt{2}}(C\Gamma^\mu)_{\alpha\gamma}$$

and similarly for the tilded objects with $z \to \bar{z}$.

This gives

$$\mathcal{A}^{(3)}_{S^2} = C_{S^2}\tilde{\delta}^{10}\left(\sum k\right)F_1^{\alpha\beta}F_2^{\gamma\delta}\zeta_{\mu\nu}(C\Gamma^\mu)_{\alpha\gamma}(C\Gamma^\nu)_{\beta\delta}|z_{12}|^{-1/2-3/2+2}|z_{23}|^{-1-1+2}|z_{31}|^{-1-1+2}\prod_{i<j}|z_{ij}|^{\alpha' k_i\cdot k}$$

where I've lumped all the path integral normalizations into one along with the vertex normalizations and the dilaton-dependence; applying the optical theorem relates these to the string coupling. Using $k_1^2 = 0, \sum k = 0$, we see that $k_i \cdot k_j = 0$, so the z-dependence dies as it must. We get

$$\mathcal{A}^{(3)}_{S^2} = C_{S^2}\tilde{\delta}^{10}\left(\sum k\right)F_1^{\alpha\beta}F_2^{\gamma\delta}\zeta_{\mu\nu}(C\Gamma^\mu)_{\alpha\gamma}(C\Gamma^\nu)_{\beta\delta}.$$

4. **The bosonic tachyon and the superstring.**

   The bosonic string tachyon, whose vertex operator is $e^{ik\cdot X}$, is not a physical state of the superstring because it isn't supersymmetric, *i.e.* it is not killed by the T_F part of the BRST operator. However, a skeptic might worry that it would somehow try to appear in interactions of allowed states (even though the BRST construction guarantees that it does not). In fact, it comes dangerously close to appearing, as follows. [1]

   Consider the OPE of two (-2)-picture NS tachyons (ignore the contribution of the superconformal ghosts here, which gives an extra factor of $e^{-2\phi}$):

$$V_k^{(-2)} \equiv ik\cdot\psi e^{ik\cdot X}.$$

[1] This problem follows some comments in section 9.13 of Polyakov's book.

Show that it contains pole terms that look very much like the bosonic tachyon vertex $e^{ik\cdot X}$. Become afraid. Then show that the coefficient of this term vanishes when the on-shell condition for the superstring tachyon is imposed.

$$V^{(-2)}_{k_1}(z)V^{(-2)}_{k_2}(0) =: ik_1 \cdot \psi(z)e^{ik_1\cdot X(z)} :: ik_2 \cdot \psi(0)e^{ik_2\cdot X(0)} :$$
$$\sim -k_1^\mu k_2^\nu \left(\frac{\delta_{\mu\nu}}{z} + z : \psi\partial\psi : +...\right) z^{\alpha' k_1\cdot k_2} : e^{ik_1\cdot X(z)+ik_2\cdot X(0)} :$$
$$\sim -k_1 \cdot k_2 \frac{1}{z^{1+\alpha' k_1\cdot k_2}} : e^{i(k_1+k_2)\cdot X(0)} : (1 + \mathcal{O}(z))$$

which looks dangerously like we've produced the bosonic tachyon, which could go on shell if its momentum $p = k_1 + k_2$ satisfied

$$\alpha' p^2 = 2. \qquad (Irma)$$

But now we must realize that the tachyons are on-shell:

$$\alpha' k_i^2 = 1, \ i = 1, 2$$

which using $(Irma)$ implies

$$2k_1 \cdot k_2 = (k_1 + k_2)^2 - k_1^2 - k_1^2 = 2 - 1 - 1 = 0,$$

and so the residue of the would-be on-shell bosonic tachyon pole is zero. Thank goodness.

The following problems don't actually require a written response:

5. **Anomalies.** Read Polchinski section 12.2 about anomalies in type I supergravity, and their cancellation using the Green-Schwarz mechanism. Look at section 12.6 where JP shows that for the heterotic string the required $B\wedge \mathrm{tr}\ F^4$ term is generated at one loop.

6. **The oracle speaks on GSO.**

   Try to understand the following statements in favor of the GSO projection, from Polyakov's book (p. 251). "*...only under the above prescription* [*i.e.* only when making the GSO projection and summing over R and NS sectors] *is it possible to treat the system in terms of spin operators* [*i.e.* is it equivalent to an Ising model]. *For that matter, take an Ising model on a surface with high genus. We know that usually* [*i.e.* on the plane] *this model can be replaced by free fermions. Is this still true? In fermionization of Ising spins a crucial role is played by Kramers-Wannier duality... Fermionic lines are essentially the boundaries of drops containing reversed spins. However, if the surface is homologically nontrivial* [*i.e.* genus > 0], *there are closed paths which do not form boundaries of anything. We must ensure that fermionic trajectories corresponding to these paths do not contribute. The way to achieve this is just to sum over spin structures, since then each homologically nontrivial path will be cancelled by one of the opposite spin structure.*"

7. **Geometry.**

   Soon we are going to start using some fancy geometry. Start looking at section 2 of hep-th/9702155.