MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics
String Theory (8.821) – Prof. J. McGreevy – Fall 2007

Solution Set 4

String scattering, open strings, supersymmetry warmup.

Reading: Polchinski, Chapter 6. Please see course webpage for supersymmetry refs.

Due: Thursday, October 25, 2007 at 11:00 AM, in lecture or in the box.

1. **Another tree amplitude (Polchinski 6.11).**

(a) For what values of $\zeta_{\mu,\nu}$ and k^μ does the vertex operator

$$\mathcal{O}_\zeta(k) = c\tilde{c}g_c'\zeta_{\mu\nu}(k) : \partial X^\mu \bar{\partial} X^\nu \ e^{ik\cdot X} :$$

create a physical state of the bosonic string ?

We showed in class that an operator of the above form creates a physical state as long as the object multiplying $c\tilde{c}$ **(let's call it** $\mathcal{V} = \zeta_{\mu\nu}(k) : \partial X^\mu \bar{\partial} X^\nu \ e^{ik\cdot X} :$**) is a conformal primary operator of weight (1,1). Let's check:**

$$T_{zz}(z)\mathcal{V}(0) = \frac{\alpha' k^2/4 + 1}{z^2}\mathcal{V}(0) + \frac{1}{z}\partial\mathcal{V}(0) + \frac{1}{z^2}\zeta_{\mu\nu}\alpha' k^\mu : \bar{\partial} X^\nu \ e^{ik\cdot X} :$$

The extra term on the RHS vanishes if

$$\zeta_{\mu\nu}k^\mu = 0,$$

and in that case $\mathcal{V}$ **is a primary of weight** $1 + \alpha' k^2/4$ **. Acting with** $T_{\bar{z}\bar{z}}$**, we require**

$$\zeta_{\mu\nu}k^\nu = 0.$$

If this condition isn't satisfied, $\mathcal{V}$ **is in fact a descendant:**

$$ik_\mu : \partial X^\mu \bar{\partial} X^\nu \ e^{ik\cdot X} : \ = L_{-1} : \bar{\partial} X^\nu \ e^{ik\cdot X} :$$

Note that this is also the condition that the polarization isn't longitudinal, *i.e.* **gauge trivial in the target space: target space gauge**

symmetry is the first of the many conditions from the Vir constratints. Also we require $k^2 = 0$, so $h = 1$.

(b) Compute the three-point scattering amplitude for a massless closed string and two closed-string tachyons at tree level.

$$\mathcal{A}_{S^2}(k_1, \zeta; k_2, k_3) = g_c^2 g_c' e^{-2\lambda} \zeta_{\mu\nu} \left\langle \mathcal{O}_\zeta(k; z_1, \bar{z}_1) : \tilde{c}ce^{ik_2\cdot X}(z_2, \bar{z}_2) :: \tilde{c}ce^{ik_3\cdot X}(z_3, \bar{z}_3) : \right\rangle_{S^2}.$$

$$\langle \prod_{i=1}^{3} c(z_i)\tilde{c}(z_i)\rangle_{S^2}^{gh} = \prod_{i<j}^{3} |z_{ij}|^2 C_{S^2}^{gh}.$$

$$\left\langle : \partial X \bar{\partial} X e^{ik_1\cdot X}(z_1, \bar{z}_1) :: e^{ik_2\cdot X}(z_2, \bar{z}_2) :: e^{ik_3\cdot X}(z_3, \bar{z}_3) : \right\rangle_{S^2}^{X} =$$

$$iC_{S^2}^X \tilde{\delta}(\sum^3 k_i) \prod_{i<j}^{3} |z_{ij}|^{\alpha' k_i \cdot k_j} \cdot \left(-\frac{i\alpha'}{2}\right)\left(\frac{k_2^\mu}{z_{12}} + \frac{k_3^\mu}{z_{13}}\right)\left(-\frac{i\alpha'}{2}\right)\left(\frac{k_2^\mu}{\bar{z}_{12}} + \frac{k_3^\mu}{\bar{z}_{13}}\right)$$

Now we play with the kinematics. The mass-shell conditions are $\alpha' k_1^2 = 0, \alpha' k_2^2 = \alpha' k_3^2 = +4$. These imply that

$$k_1 \cdot k_2 = k_1 \cdot k_3 = 0, \alpha' k_2 \cdot k_3 = 4.$$

Using this and the relation $C_{S^2} = C_{S^2}^{gh} C_{S^2}^X e^{-2\lambda} = \frac{8\pi}{\alpha' g_c^2}$, and the condition derived in part (a), $\zeta_{\mu\nu} k_1^\mu = 0$, the 3-point amplitude can therefore be written as

$$\mathcal{A} = -i2\pi\alpha' g_c' \tilde{\delta}(\sum k)\zeta_{\mu\nu} \left|\frac{z_{12}z_{13}}{z_{23}}\right|^2 \left(\frac{k_2^\mu}{z_{12}} + \frac{k_3^\mu}{z_{13}}\right)\left(\frac{k_2^\mu}{\bar{z}_{12}} + \frac{k_3^\mu}{\bar{z}_{13}}\right)$$

$$= -\frac{\pi i\alpha'}{2} g_c' \tilde{\delta}^{26}\left(\sum k\right) \zeta_{\mu\nu} k_{23}^\mu k_{23}^\nu.$$

(c) Factorize the tachyon four-point amplitude on the massless pole (in say the s-channel), and use unitarity to relate the massless coupling g_c' to g_c, the coupling for the tachyon.

The Virasoro amplitude is

$$\mathcal{A}_{S^2}^{(4)}(k_1, ..k_4) = \frac{8\pi i g_c^2}{\alpha'} \tilde{\delta}^{26}(\sum^4 k) \int_{\mathbb{C}} d^2 z_4 |z_4|^{-\frac{\alpha' u}{2}-2} |1 - z_4|^{-\frac{\alpha' t}{2}-2}$$

The massless s-channel pole comes from the next-to-leading term in the expansion of the integrand near $z_4 \to \infty$; alternatively, we can see it from properties of gamma functions as follows.

$$\mathcal{A}^{(4)}_{S^2} = \frac{8\pi i g_c^2}{\alpha'}\tilde{\delta}^{26}(\sum^4 k)2\pi\frac{\Gamma\left(-1-\frac{\alpha' s}{4}\right)\Gamma\left(-1-\frac{\alpha' t}{4}\right)\Gamma\left(-1-\frac{\alpha' u}{4}\right)}{\Gamma\left(-2-\frac{\alpha' s}{4}\right)\Gamma\left(-2-\frac{\alpha' t}{4}\right)\Gamma\left(-2-\frac{\alpha' u}{4}\right)}$$

Near $x = -1$, $\Gamma(x) \sim -\frac{1}{x+1}$, so the massless s-channel pole (where $\alpha'(t+u) \sim -16$) comes from

$$\Gamma\left(-1-\frac{\alpha' s}{4}\right) \overset{s\sim 0}{\simeq} \frac{4}{\alpha' s}.$$

Using $\Gamma(x+1) = x\Gamma(x)$, the rest of the gammas give:

$$\frac{1}{\Gamma(2)}\frac{\Gamma\left(-1-\frac{\alpha' t}{4}\right)\Gamma\left(-1-\frac{\alpha' u}{4}\right)}{\Gamma\left(-2-\frac{\alpha' t}{4}\right)\Gamma\left(-2-\frac{\alpha' u}{4}\right)} = \left(-2-\frac{\alpha' t}{4}\right)\left(-2-\frac{\alpha' u}{4}\right) = -4+\left(\frac{\alpha'}{2}\right)^2 tu.$$

We will later realize that we should rewrite this using $-\frac{16}{\alpha'} = (u+t)^2 = (u-t)^2 + 4ut$ as

$$-4 + \left(\frac{\alpha'}{2}\right)^2 tu = -\left(\frac{\alpha'}{4}\right)^2 (u-t)^2.$$

So the four point factorizes as

$$\mathcal{A}^{(4)}(\alpha' s \sim 0) \sim \frac{16\pi^2 i g_c^2}{\alpha'}\tilde{\delta}^{26}(\sum^4 k)\left(\frac{-\left(\frac{\alpha'}{4}\right)^2 (u-t)^2 .}{(-\alpha' s/4)}\right) \quad . \qquad (Elmo)$$

The optical theorem says

$$\mathcal{A}_{S^2}(k_1..k_4) = i\int \frac{d^{26}k}{(2\pi)^{26}}\sum_\zeta \frac{\mathcal{A}_{S^2}(k_1,k_2;k,\zeta)\mathcal{A}_{S^2}(-k,\zeta;k_3,k_4)}{-k^2+i\epsilon} + \text{reg. at } s=0.$$

Using the above expression for $\mathcal{A}^{(3)}$, this is

$$\mathcal{A}_{unitarity} = -i\sum_\zeta \frac{\pi^2(\alpha')^2}{4}(g_c')^2\tilde{\delta}(\sum k)\frac{\zeta_{\mu\nu}k_{12}^\mu k_{12}^\nu \zeta^\star_{\rho\lambda}k_{34}^\rho k_{34}^\lambda}{s+i\epsilon}$$

The sum over polarizations ζ runs over normalized $\zeta_{\mu\nu}\zeta^{\mu\nu} = 1$ two-tensors which are transverse to the momentum of the intermediate state $k = k_1 + k_2$:

$$\zeta_{\mu\nu}k^{\mu} = 0 = \zeta_{\mu\nu}k^{\nu}.$$

One basis element for the space of such tensors is

$$e_{\mu\nu} = \frac{k_{12\mu}k_{12\nu}}{k_{12}^2}$$

which is transverse because $k_1^2 = k_2^2$. The sum over polarizations is

$$S \equiv \sum_{\zeta} \zeta_{\mu\nu}k_{12}^{\mu}k_{12}^{\nu}\zeta^{\star}_{\rho\lambda}k_{34}^{\rho}k_{34}^{\lambda}$$

In fact none of the other basis elements contribute, and we get

$$S = (k_{12} \cdot k_{34})^2 .$$

Alternatively, just as the sum over polarizations of photons (see *e.g.* **Srednicki p.358) can be replaced with**

$$\sum_{\epsilon} \epsilon_{\mu}(k)\epsilon^{\star}_{\nu}(k) \to \eta_{\mu\nu}$$

when appearing in gauge-invariant amplitudes (the terms that are missing on the LHS don't couple by gauge invariance), here we would find similar rule of the form

$$\sum_{\zeta} \zeta_{\mu\nu}\zeta^{\star}_{\rho\sigma} \to \eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho},$$

which reproduces the result above.

Some kinematical voodoo gives

$$S = (k_{12} \cdot k_{34})^2 = (k_1 \cdot k_3 - k_2 \cdot k_3 + k_2 \cdot k_4 - k_1 \cdot k_4)^2 = (u-t)^2.$$

This leaves us with

$$\mathcal{A}_{unitarity} = -i\frac{\pi^2(\alpha')^2}{4}(g_c')^2\tilde{\delta}(\sum k)\frac{(u-t)^2}{s+i\epsilon} \qquad (Kermit)$$

Comparing (*Elmo*) and (*Kermit*) leads to the relation

$$g_c' = \frac{2}{\alpha'} g_c,$$

which lets us eliminate g_c'. Such a relation exists for every closed string state.

2. **High-energy scattering in string theory.**[1]

Consider the tree-level scattering amplitude of N bosonic string states, with momenta k_i^μ:

$$\mathcal{A}^{(n)}(\{k_1, ..., k_n\}) = \int \prod_{i=4}^{n} d^2 z_i \left\langle \prod_{i=1}^{n} \mathcal{V}_{k_i}(z_i, \bar{z}_i) \right\rangle_{S^2}.$$

In this problem we will study the limit of hard scattering (also called fixed-angle scattering), where we scale up all the momenta uniformly,

$$k_i^\mu \mapsto \alpha k_i^\mu, \quad \alpha \to \infty;$$

in terms of the Lorentz-invariant Mandelstam variables, we are taking the limit $s_{ij} \to \infty$, s_{ij}/s_{kl} fixed. It was claimed in class that the 4-point function behaves in this limit like $e^{-s\alpha' f(\theta)}$ where θ is the scattering angle. In this limit, $k^2 \gg m^2$ (if both aren't zero) and we can ignore the parts of the vertex operators other than $e^{ik\cdot X}$.

(a) Notice that the integral over X is always gaussian, hence the action of the saddle point solution gives the exact answer. Find the saddle-point configuration of $X(z)$ and evaluate the on-shell action.

$$\mathcal{A}^{(n)}(\{k_1, ..., k_n\}) = \int \prod_{i=4}^{n} d^2 z_i \left\langle \prod_{i=1}^{n} : e^{ik_i\cdot X}(z_i, \bar{z}_i) : \right\rangle_{S^2}.$$

$$\left\langle \prod_{i=1}^{n} : e^{ik_i\cdot X}(z_i, \bar{z}_i) : \right\rangle_{S^2} = \int [DX]\ e^{-\frac{1}{2\pi\alpha'}\int d^2 z \left(\partial X\cdot\bar{\partial} X + J\cdot X\right)}$$

with

$$J^\mu = -2\pi\alpha' i \sum_i k_i^\mu \delta^2(z - z_i).$$

[1] This discussion follows the papers of Gross and Mende.

Complete the square:

$$\partial X \bar{\partial} X + J \cdot X = \partial \tilde{X} \bar{\partial} \tilde{X} - \frac{1}{4} J^2$$

with

$$\tilde{X}(z) \equiv X(z) + \frac{1}{2} \int d^2 z' G(z, z') J(z')$$

where G is the Green function: $\partial \bar{\partial} G = \delta^2$. Under this linear field redefinition, the measure is $[D\tilde{X}] = [DX]$ and we find

$$\left\langle \prod_{i=1}^{n} : e^{ik_i \cdot X}(z_i, \bar{z}_i) : \right\rangle_{S^2} = e^{-\frac{1}{2\pi\alpha'} \int d^2 z \frac{J^2}{4}} Z \quad (Fred)$$

where

$$Z = \int [D\tilde{X}] \; e^{-\frac{1}{2\pi\alpha'} \int d^2 z \left(\partial \tilde{X} \cdot \bar{\partial} \tilde{X} \right)}$$

is just a number; the saddle point for $\tilde{X}$ is at $\tilde{X} = 0$, which means the saddle for X is at

$$X_\star = -\frac{1}{2} \int d^2 z' G(z, z') J(z').$$

The Green function on the sphere is

$$G(z, z') = -\frac{1}{2\pi} \ln |z - z'|^2$$

so plugging in our expression above for J we get

$$X_\star^\mu = \frac{1}{4\pi} \int d^2 z' \ln |z - z'|^2 J(z')^\mu = -i \frac{\alpha'}{2} \sum_{i=1}^{n} k_i^\mu \ln |z - z_i|^2.$$

Notice that it's imaginary! Crazy! Nevertheless we'll see that it gives the correct answer. Another expression which is sometimes useful is

$$X_\star^\mu(z) = -\frac{1}{2} \int d^2 z' G(z, z') J(z')^\mu = \pi i \alpha' \sum_{i}^{n} G(z, z_i) k_i^\mu \qquad (Eldridge).$$

The on-shell action is just minus the thing in the exponent in $(Fred)$:

$$S_\star(k) = \frac{1}{2\pi\alpha'} \int d^2 z \frac{J^2}{4} = \frac{1}{2\pi\alpha'} \int d^2 z \left(-X_\star \partial \bar{\partial} X_\star + J X_\star \right).$$

Using the saddle point equation $\partial\bar{\partial}X = \frac{1}{2}J$, this is

$$S_\star(k) = \frac{1}{4\pi\alpha'}\int d^2z J X_\star = -\frac{i}{2}\sum_i k_i \cdot X_\star(z_i)$$

When we plug in the expression for the saddle point we will get terms of the form $k_i^2 \ln|z_i - z_i|$. These are exactly the terms we subtract when we define the normal-ordered operators, and we ignore them for this reason. With this in mind, we find:

$$S_\star(k) = -\frac{\alpha'}{2}\sum_{i<j} k_i \cdot k_j \ln|z_i - z_j|^2 \quad .$$

(b) In the hard-scattering limit, the integral over the positions of the $n-3$ unfixed vertex operators is also well-described by a saddle-point approximation. Convince yourself that this is true; *i.e.* that the saddle point is well-peaked.

We want to study the saddle point of the integrals over z_i in

$$\mathcal{A}^{(n)} = \int \prod_{i=4}^n d^2 z_i e^{-S_\star(k)} = \int \prod_{i=4}^n d^2 z_i \prod_{i<j} |z_{ij}|^{-\frac{\alpha'}{2}k_i \cdot k_j}.$$

It's useful to rewrite this using the definition $s_{ij} \equiv (k_i + k_j)^2$ and the mass-shell condition $k_i^2 = -m_i^2$:

$$k_i \cdot k_j = -\frac{1}{2}(s_{ij} - m_i^2 - m_j^2) \sim -\frac{1}{2}s_{ij}$$

where we used the hard-scattering approximation in the last step. The action for z_i in $\int dz e^{-S(z)}$ is then

$$S(z) = \frac{\alpha'}{4}\sum_{i<j} s_{ij} \ln|z_{ij}|^2.$$

The basic idea is that scaling up all the Mandelstam variables $s_{ij} \to \alpha s_{ij}$ scales up the action

$$S(z) \to \alpha S(z)$$

and sharpens any features it has, exactly like the $\hbar \to 0$ limit in QM. More precisely, the condition for the saddle point

$$0 = \partial_{z_i} S$$

has solutions $z^\star$ which don't scale with α, but the masses of fluctuations at the saddle

$$\partial_i \partial_j S|_{z^\star}$$

scale like α, making the van vleck determinant negligible as $\alpha \to \infty$.

(c) For the four-point function, find the saddle-point for the z_4 integral and evaluate the action on the saddle. Compare to the claimed behavior of the exact tree-level answer.

Using $\ln a + \ln b = \ln ab$, the action for the z_4 integral in the four-point function is

$$S(z) = \frac{\alpha'}{2} \left(s \ln |z_{12} z_{34}| + t \ln |z_{13} z_{24}| + u \ln |z_{14} z_{23}| \right)$$

which using $s = -t - u$ (ignoring masses) is

$$S(z) = \frac{\alpha'}{2} \left(t \ln \left| \frac{z_{13} z_{24}}{z_{12} z_{34}} \right| + u \ln \left| \frac{z_{14} z_{23}}{z_{12} z_{34}} \right| \right) = \frac{\alpha'}{2} \left(t \ln |\lambda| + u \ln |1 - \lambda| \right)$$

with the cross-ratio $\lambda \equiv \frac{z_{13} z_{24}}{z_{12} z_{34}}$

$$\frac{\partial S(\lambda)}{\partial \lambda} \propto \frac{t}{\lambda} + \frac{u}{1 - \lambda}$$

whose solution is

$$0 = t(1 - \lambda_\star) - u \lambda_\star \Longrightarrow \lambda_\star = \frac{t}{u + t} = -\frac{t}{s}.$$

Plugging back into the action, we find

$$S(\lambda_\star) = \frac{\alpha'}{2} \left(t \ln |t/s| + u \ln |\frac{s+t}{s}| \right) = \frac{\alpha'}{2} \left(s \ln s + t \ln t + u \ln u \right).$$

And we get

$$A^{(4)} \sim e^{-S(\lambda_\star(k))} = e^{\frac{\alpha'}{2}(s \ln \alpha' s + t \ln \alpha' t + u \ln \alpha' u)} \quad .$$

This is exactly what results from using Stirling's formula on the gamma functions in the exact tree amplitude. (Note that the α's in the logs cancel between the three terms and we could have put any scale there we wanted.) Notice that all the factors of i in the saddle point configuration for X resulting in a real action which *suppresses* the amplitude. This process doesn't happen very often!

3. **Open string boundary conditions.** Polchinski Problems 1.6 and 1.7.

   I don't want to do this one. You all got it exactly right.

4. **T-duality: not just for the free theory.** Polchinski Problem 8.3.

   Consider the sigma model whose action (in conformal gauge) is

$$S(\partial X, Y) = S(Y) + \frac{1}{4\pi\alpha'} \int d^2z \left(\delta^{ab} G_{XX}(Y) \partial_a X \partial_b X + \left(\delta^{ab} G_{\mu X} + \epsilon^{ab} B_{\mu X}\right) \partial_a X \partial_b Y^\mu\right) \quad .$$

   Here Y^μ are a bunch of coordinates on which the background fields depend in arbitrarily complicated ways. X only appears through its derivatives.

   Now to gauge this shift symmetry, we add a 2d gauge field A and replace everywhere $\partial_a X \longrightarrow \partial_a X + A_a$, so that everything will be invariant under local shifts

$$X \to X + \lambda(z), A_a \to A_a - \partial_a \lambda.$$

   Imagine I've typed this.

   In order not to actually change anything, we also add a Lagrange multiplier $\hat{X}$ that kills the gauge field we've just added:

$$S_\theta \equiv i \int d^2z \hat{X} F = i \int d^2z \hat{X} \epsilon^{ab} \partial_a A_b,$$

   where $F = dA$. $\hat{X}$ couples like an axion, a dynamical theta-angle, multiplying the otherwise-topological density F. Integrating over $\hat{X}$,

$$\int [Dhx] e^{i \int d^2z \hat{X} \epsilon^{ab} \partial_a A_b} = \delta[F]$$

   On a topologically trivial worldsheet, $F = 0$ says by the Poincare lemma that A is pure gauge. In Lorentz gauge we learn that A is constant, which just amounts to a constant shift of X, which we can be absorbed by a field redefinition, and we recover the original theory. The claim here is

$$e^{-S(\partial X, Y)} = \int \frac{[D\hat{X} DA]}{\mathrm{Vol}(\mathcal{G})} \; e^{-S(\partial X + A, Y) + S_\theta(A, \hat{X})}.$$

If instead we use the gauge symmetry to fix $X = 0$, we find the action

$$S(Y)+\frac{1}{4\pi\alpha'}\int d^2z\left(\delta^{ab}G_{XX}(Y)A_aA_b+\left(\delta^{ab}G_{\mu X}+\epsilon^{ab}B_{\mu X}\right)A_a\partial_bY^\mu-i\partial_a\hat{X}\epsilon^{ab}A_b\right) \quad .$$

In the last step here I've integrated by parts the theta term. The A integral is gaussian! Let's do it. (Notice that the action for Y just goes along for the ride here.) The important bit is:

$$L \equiv G_{XX}A^2 + A_a(\partial_bYE^{ab} - i\partial_b\hat{X}\epsilon^{ab})$$

where

$$\partial_bYE^{ab} \equiv \partial_bY^\mu\left(\delta^{ab}G_{\mu X}+\epsilon^{ab}B_{\mu X}\right).$$

Notice that the $\hat{X}$ term contributes in the same way as the $B_{X\mu}$ field here. Let's complete the square:

$$L = G_{XX}\left(A+\frac{1}{2}G^{XX}\left(E^{ab}\partial_bY - i\epsilon^{ab}\partial_b\hat{X}\right)\right)^2 - \frac{G^{XX}}{4}\left(E^{ab}\partial_bY - i\epsilon^{ab}\partial_b\hat{X}\right)\left(E^{bc}\partial_cY - i\epsilon^{bc}\partial_c\hat{X}\right)$$

The A integral now looks like

$$\int [DA]\ e^{\frac{1}{4\pi\alpha'}\int d^2zL} = e^{-\int J^2}\int [DA]\ e^{-\int d^2z\frac{G_{XX}}{4\pi\alpha'}\tilde{A}^2}$$

with

$$J^2 \equiv -\frac{G^{XX}}{4}\left(E^{ab}\partial_bY - i\epsilon^{ab}\partial_b\hat{X}\right)\left(E^{bc}\partial_cY - i\epsilon^{bc}\partial_c\hat{X}\right)$$

and $\tilde{A} \equiv A+\frac{1}{2}G^{XX}\left(E^{ab}\partial_bY - i\epsilon^{ab}\partial_b\hat{X}\right)$. We can change variables $[DA] = [D\tilde{A}]$ for free, and we just end up with

$$e^{-\int J^2}\int [D\tilde{A}]\ e^{-\int d^2z\frac{G_{XX}}{4\pi\alpha'}\tilde{A}^2} = e^{-\int J^2}\det{}^{-1}\frac{G_{XX}}{2\alpha'}.$$

The determinant looks stupid but actually generates a correction to the dilaton profile of the form $\hat{\Phi} = \Phi - \ln G_{XX}$; this is necessary for T-duality to preserve the strength of gravity in the non-compact directions.

The J^2 term corrects the action for Y and $\hat{X}$ in an important way. In particular, it generates a *kinetic term* for $\hat{X}$, which was previously a lowly Lagrange multiplier. In particular, if $G_{X\mu} = 0 = B_{X\mu}$,

$$\int J^2 = \int d^2z\left(-i^2\right)\frac{G^{XX}}{4\pi\alpha'}\epsilon^{ab}\epsilon^{bc}\partial_a\hat{X}\partial_c\hat{X} = +\frac{1}{4\pi\alpha'}\int d^2zG^{XX}\partial_a\hat{X}\partial^a\hat{X}$$

which is a positive kinetic term, and we see that $\hat{X}$ has the opposite radius from X: $G^{XX} = \frac{1}{G_{XX}}$.

More generally, if there is some KK gauge field $G_{X\mu}$ it leads to a nonzero B field in the T-dual:

$$\hat{B}_{\hat{X}\mu} = G^{XX} G_{X\mu} \quad !$$

(hats denote quantities in the T-dualized theory). Similarly, a B field with one leg along the T-dualized circle leads to a KK gauge field:

$$\hat{G}_{\hat{X}\mu} = G^{XX} B_{X\mu} \quad !$$

This is exactly what we should expect given that we are interchanging momentum (the charge to which the KK gauge field $G_{X\mu}$ couples) and string winding (the charge to which the AS tensor component $B_{X\mu}$ couples). There are also terms in J^2 which correct the metric and AS tensor for Y:

$$G_{\mu\nu} \mapsto \hat{G}_{\mu\nu} = G_{\mu\nu} - G^{XX} G_{X\mu} G_{X\nu} + G^{XX} B_{X\mu} B_{X\nu}.$$

$$B_{\mu\nu} \mapsto \hat{B}_{\mu\nu} = B_{\mu\nu} - G^{XX} G_{X\mu} B_{X\nu} + G^{XX} B_{X\mu} G_{X\nu}.$$

The following problems are intended to get everyone up to speed with supersymmetry and its consequences. They are more optional than the other ones.

1. **Supersymmetric point particle.**

   Consider the action for a spinning particle

$$S = \int d\tau \left(-\frac{\dot{x}^2}{2e} + e^{-1} i \dot{x}^\mu \psi_\mu \chi - i \psi^\mu \dot{\psi}_\mu \right).$$

   The ψ^μs are real grassmann variables, fermionic analogs of x^μ, which satisfy

$$\psi^\mu \psi^\nu = -\psi^\nu \psi^\mu.$$

   (a) Show that this action is reparametrization invariant, *i.e.*

$$\delta x^\mu = \xi \dot{x}^\mu, \quad \delta \psi^\mu = \xi \dot{\psi}^\mu$$

$$\delta e = \partial_\tau (\xi e), \quad \delta \chi = \partial_\tau (\xi \chi)$$

is a symmetry of S.

(b) Show that the (local) *supersymmetry* transformation

$$\delta x^\mu = i\psi^\mu \epsilon, \quad \delta e = i\chi\epsilon$$

$$\delta\chi = \dot{\epsilon}, \quad \delta\psi^\mu = \frac{1}{2e}(\dot{x}^\mu + i\chi\psi^\mu)\epsilon.$$

with ϵ a grassmann variable, is a symmetry of S.

(c) Consider the gauge $e = 1, \chi = 0$. What is the equation of motion for χ? What familiar equation for ψ do we get?

2. **Strings in flat space with worldsheet supersymmetry.**

Consider the action for D free bosons and D free fermions in two dimensions:

$$S = -\frac{1}{2\pi}\int d^2\sigma \;\left(\partial_a X^\mu \partial^a X_\mu - i\bar{\psi}^\mu \rho^a \partial_a \psi_\mu\right).$$

The spacetime $\mu = 0..D-1$ indices are contracted with $\eta_{\mu\nu}$. Here ψ^μ are 2d two-component majorana spinors, and ρ^a are 2d 'gamma' matrices, *i.e.*, they participate in a 2d Clifford algebra $\{\rho^a, \rho^b\} = -2\eta^{ab}$ (we'll work on a Lorentzian worldsheet for this problem), and $\bar{\psi} \equiv \psi^\dagger \rho^0$. Pick a basis for the 2d 'gamma' matrices of the form

$$\rho^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \rho^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \quad (1)$$

(a) Show that the fermion part of the action above leads to the massless Dirac equation for the worldsheet fermions ψ. Show that with the chosen basis of gamma matrices ρ^α, the Dirac equation implies that $\psi_\pm$ is a function of $\sigma^\pm$ only (where I'm letting the indices on $\rho^a_{\alpha\beta}$ run over $\alpha, \beta = \pm$).

The equation of motion for ψ results from

$$0 = \delta_\psi S \propto 2\int \bar{\delta}\psi \rho^a \partial_a \psi$$

where the two comes from the fact that we get the same term from varying both ψs after IBP. So the EOM is just the 2d Dirac equation

$$0 = \rho^a \partial_a \psi.$$

In the basis above, this is

$$0 = \left(\begin{pmatrix} 0 & -i\partial_0 \\ i\partial_0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & i\partial_1 \\ i\partial_1 & 0 \end{pmatrix} \right) \psi = i \begin{pmatrix} 0 & (-\partial_0 + \partial_1) \\ (\partial_0 + \partial_1) & 0 \end{pmatrix} \begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix} = i \begin{pmatrix} \partial_- \psi_+ \\ \partial_+ \psi_- \end{pmatrix}.$$

Next we want to show that the (global) *supersymmetry transformation*

$$\delta X^\mu = \bar{\epsilon}\psi^\mu$$

$$\delta\psi^\mu = -i\rho^a \partial_a X^\mu \epsilon$$

is a symmetry of the action S.

(b) A useful first step is to show that for Majorana spinors

$$\bar{\chi}\psi = \bar{\psi}\chi.$$

Since for Majorana spinors, $\bar{\chi} = \chi^T \rho^0$, we have $\bar{\chi}\psi = \chi^T \rho^0 \psi$. But ρ^0 is an antisymmetric matrix, so rearranging the indices costs a minus sign, but so does interchanging the two grassmann variables, and we're out.

(c) Show that the action S is supersymmetric.

Part (b) implies that

$$\delta\bar{\psi}^\mu = (\delta\psi^\mu)^T \rho^0 (\delta\psi^\mu)^\dagger \rho^0 = +i\partial_a X^\mu \epsilon^\dagger (\rho^a)^\dagger \rho^0 = i\partial_a X^\mu \bar{\epsilon}\rho^a$$

where I've used the slightly nontrivial but necessary fact that

$$\bar{\rho}^a \equiv (\rho^a)^\dagger \rho^0 = \rho^0 \rho^a, \quad a = 0, 1.$$

Armed with this, the variation of the lagrangian is

$$\delta(\partial_a X \partial^a X - i\bar{\psi}\rho^a \partial_a \psi) = 2\partial_a(\bar{\epsilon}\psi)\partial^a X - i(i\bar{\epsilon}\partial_b X \rho^b)\rho^a \partial_a \psi - i\bar{\psi}\rho^a \partial_a(-i\rho^b \partial_b X \epsilon) \equiv A + B + C$$

$$A = 2\partial_a(\bar{\epsilon}\psi\partial^a X) - 2\bar{\epsilon}\psi\partial^2 X$$

where the first term is a total derivative which we can forget.

$$B = (-i)^2 \partial_a(\bar{\epsilon}\partial_b X \rho^b)\rho^a \psi + t.d. = -\partial_a \bar{\epsilon}(\partial_b X \rho^b \rho^a \psi) - \bar{\epsilon}\partial_a \partial_b X \rho^b \rho^a \psi.$$

The first term here I'm keeping for part (d), and the second, using the clifford algebra, is

$$-\bar{\epsilon}\partial_a\partial_b X\rho^b\rho^a\psi = -\frac{1}{2}\bar{\epsilon}\partial_a\partial_b X(-2\eta^{ab})\psi = +\bar{\epsilon}\partial^2 X\psi.$$

Similarly,

$$C = (-i)^2\bar{\psi}\rho^a\rho^b\partial_a\partial_b X\epsilon + (-i)^2\bar{\psi}\rho^a\rho^b\partial_b X\partial_a\epsilon = +\bar{\psi}\partial^2 X\epsilon - \bar{\psi}\rho^a\rho^b\partial_b X\partial_a\epsilon.$$

Altogether,

$$\delta L = t.d. - 2\bar{\epsilon}\psi\partial^2 X + \bar{\epsilon}\partial^2 X\psi\bar{\psi}\partial^2 X\epsilon - \partial_a\bar{\epsilon}(\partial_b X\rho^b\rho^a\psi) - \bar{\psi}\rho^a\rho^b\partial_b X\partial_a\epsilon.$$

The terms without derivatives of epsilon are

$$-2\bar{\epsilon}\psi\partial^2 X + \bar{\epsilon}\psi\partial^2 X + \bar{\psi}\epsilon\partial^2 X = 0$$

using the result of (b).

If ϵ is a constant, therefore, $\delta S = 0$.

(d) What is the conserved Noether current associated to the supersymmetry?

From above, the terms proportional to derivatives of epsilon in the variation of L are

$$\delta L = t.d. + -\partial_a\bar{\epsilon}(\partial_b X\rho^b\rho^a\psi) - \bar{\psi}\rho^a\rho^b\partial_b X\partial_a\epsilon.$$

Using the Majorana property, these two terms are the same and we find

$$\delta S = -\frac{1}{2\pi}\int d^2z\partial_a\bar{\epsilon}(-2)\partial_b X\rho^b\rho^a\psi \equiv -\frac{1}{\pi}\int d^2z\partial_a\bar{\epsilon}(T_F)^a.$$

So we have

$$(T_F)_a = -\frac{1}{2}\rho^b\rho_a\psi^\mu\partial_b X_\mu.$$

I'm not sure about the overall factor.

Note that the 2d identity $\rho^a\rho^b\rho_a = 2\eta^{ab}\rho_a - \rho^b\rho^a\rho_a = \rho^b - \rho^b = 0$ implies that

$$0 = \rho^a(T_F)_a$$

which is the superpartner of the statement that $T^a_a = 0$, *i.e.* is a consequence of superconformal symmetry.

(e) [Optional] Show that the algebra enjoyed by the Noether supercharges $Q_\alpha \equiv \int d\sigma G_{0\alpha}$ under Poisson brackets (or canonical (anti)-commutators) is of the form

$$\{Q_\alpha, \overline{Q}_\beta\} = -2i\rho^a_{\alpha\beta} P_a \qquad (2),$$

where P_a is the momentum.

Or equivalently, show that the commutator of two supersymmetry transformations (acting on any field) acts as a spacetime translation:

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}]\mathcal{O} = A^a \partial_a \mathcal{O}$$

where A is a constant writeable in terms of $\epsilon_{1,2}$.

The easiest thing to do is just vary twice:

$$\delta_{\epsilon_1}\delta_{\epsilon_2} X = \delta_{\epsilon_1}(\bar{\epsilon}_2 \psi) = -i\bar{\epsilon}_2 \rho^a \partial_a X \epsilon_1.$$

Switching labels,

$$\delta_{\epsilon_2}\delta_{\epsilon_1} X = -i\bar{\epsilon}_1 \rho^a \partial_a X \epsilon_2.$$

The majorana property $\bar{\chi}\psi = \bar{\psi}\chi$ with $\bar{\chi} = \overline{i\rho^a \epsilon_1}$ says

$$\bar{\epsilon}_2 i \rho^a \epsilon_1 = \overline{i\rho^a \epsilon_1}\epsilon_2 = -i\bar{\epsilon}_1 \rho^a \epsilon_2$$

and we find

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}] X = 2i\bar{\epsilon}\rho^a \epsilon_2 \partial_a X.$$

A similar expression holds for

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}]\psi$$

upon using the EOM. In general

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}]\mathcal{O} = v^a \partial_a \mathcal{O} = -iv^a P_a \mathcal{O}$$

with $v^a = 2i\bar{\epsilon}_1 \rho^a \epsilon_2$. Since the LHS of this equation is

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}]\mathcal{O} = [\bar{\epsilon}_1 Q, \bar{\epsilon}_2 Q]\mathcal{O} = \bar{\epsilon}_1^\alpha \epsilon_2^\beta \{Q_\alpha, \bar{Q}_\beta\}\mathcal{O},$$

(using the grassmann property of ϵ!) we have (using the form of v^a above

$$\{Q_\alpha, \bar{Q}_\beta\} = -2\rho^a_{\alpha\beta} P_a.$$

So I was wrong about the i.

(f) Show that the algebra (2) above implies that a state is a supersymmetry singlet ($Q|\psi\rangle = 0$) if and only if it is a ground state of H.

The basic idea is that for any state ψ,

$$-2\rho^a_{\alpha\beta}\langle\psi|P_a|\psi\rangle = \langle\psi|\{Q_\alpha\bar{Q}_\beta\}|\psi\rangle = \begin{pmatrix} 0 & -2i\langle\psi|Q_-^2|\psi\rangle \\ 2i\langle\psi|Q_+^2|\psi\rangle & 0 \end{pmatrix}.$$

The LHS is

$$-2\rho^a_{\alpha\beta}\langle\psi|P_a|\psi\rangle = \begin{pmatrix} 0 & 2i\langle\psi|(-P_0+P_1)|\psi\rangle \\ 2i\langle\psi|(P_0+P_1)|\psi\rangle & 0 \end{pmatrix} \quad .$$

The basic thing we've just learned is an algebra statement:

$$Q_\pm^2 = P_0 \pm P_1 \quad ;$$

note that the BHS is a hermitean operator, so we have the factors of is right now. So we have

$$0 \leq ||Q_+|\psi\rangle||^2 = \langle\psi|Q_+^2|\psi\rangle = \langle\psi|(P_0+P_1)|\psi\rangle$$

Similarly,

$$0 \leq ||Q_-|\psi\rangle||^2 = \langle\psi|(P_0-P_1)|\psi\rangle.$$

Adding the two inequalities (which preserves truthiness, while subtracting does not), we get

$$0 \leq 2\langle\psi|P_0|\psi\rangle = 2E_\psi$$

So the energy is positive semi-definite in a supersymmetric theory. This inequality is saturated only if one or more of the Qs kills ψ. In fact, in this case, since the ground state is (usually) translationally invariant, $P_1|0\rangle = 0$, both supercharges must kill the $E = 0$ ground state, if it exists.

(g) **(1,1) superspace.** Show that the action above can be rewritten as

$$\int d^2z \int d\theta^+ d\theta^- \; D_+\mathbf{X}\cdot D_-\mathbf{X}$$

where $\theta^\pm$ are real grassmann coordinates on $2d, \mathcal{N} = (1,1)$ superspace (*i.e.* there is one real right-moving supercharge and one real left-moving supercharge), and the (1,1) superfield is

$$\mathbf{X}(z,\bar{z},\theta^+,\theta^-) \equiv X + \theta^-\psi_- + \theta^+\psi_+ + \theta_+\theta_-F_{+-},$$

and

$$D_+ = \frac{\partial}{\partial \theta^+} + \theta^+ \frac{\partial}{\partial \sigma^-}, \qquad D_- = \frac{\partial}{\partial \theta^-} + \theta^- \frac{\partial}{\partial \sigma^+}$$

are (1,1) superspace covariant derivatives. Please note that the $\pm$ indices on the θs are 2d spin, *i.e.* sign of charge under the 2d $SO(2) \sim U(1)$ of rotations; so for example the object $D_\pm$ has spin $\pm 1/2$. The conservation of this quantity is a useful check on the calculation.

$$D_+ X = \psi_+ + \theta^- F + i\theta^+ \partial_+ X - i\theta^- \theta^+ \partial_+ \psi_-$$
$$D_- X = \psi_- - \theta^+ F + i\theta^+ \partial_+ X + i\theta^- \theta^+ \partial_- \psi_+$$

The superspace integral takes the coefficient of $\theta^- \theta^+$, which for $D_+ X \cdot D_- X$ is

$$\partial_+ X \partial_- X - F^2 + i\psi_+ \partial_- \psi_+ + i\psi_- \partial_+ \psi_- .$$

The EOM for F is $F = 0$ and we recover the action from part (a) with the given basis of gamma matrices, if we set

$$S = -\frac{1}{2\pi} \int d\theta^+ d\theta^- D_+ X D_- X.$$

3. **$\mathcal{N} = (0,2)$ and $\mathcal{N} = (2,2)$ supersymmetry.**

   In the previous problem we studied a system with $(1,1)$ supersymmetry. Many interesting theories have extended supersymmetry in two dimensions. A particularly interesting and familiar case is $2d, \mathcal{N} = (2,2)$ supersymmetry, which arises by dimensional reduction from (the conceivably realistic) $4d, \mathcal{N} = 1$ supersymmetry. [2]

   (a) Consider the action

$$\int d^2 z \int d^2 \theta \ \bar{D} \mathbf{X} D \mathbf{X}$$

   where now θ is a *complex* grassmann variable, $d^2\theta = d\theta d\bar{\theta}$,

$$\mathbf{X}(z, \bar{z}, \theta, \bar{\theta}) \equiv X + \theta\psi + \bar{\theta}\bar{\psi} + \theta\bar{\theta} F$$

   are 2d $\mathcal{N} = 2$ superfields and

$$D = \frac{\partial}{\partial \theta} + \bar{\theta} \partial_z, \qquad \bar{D} = \frac{\partial}{\partial \bar{\theta}} + \theta \bar{\partial}_{\bar{z}}$$

[2] For this problem, we return to a euclidean worldsheet.

are $\mathcal{N} = 2$ superspace covariant derivatives. What does this look like in components?

Note that I've reverted to the definition of $D, \bar{D}$ in the original version of the pset. I did this because now D has a definite charge under the $U(1)$ (R)-symmetry under which $\theta \to e^{i\alpha}\theta$. Using this,

$$DX = \psi + \bar{\theta}(F + \partial X) - \theta\bar{\theta}\partial\psi$$

$$\bar{D}X = \bar{\psi} + \theta(-F + \partial X) + \theta\bar{\theta}\partial\psi$$

Not too surprisingly, this gives

$$DX\bar{D}X_{\theta\bar{\theta}} = -\partial X\bar{\partial}X + 2\bar{\psi}\bar{\partial}\psi + F^2$$

Note that the bar on the psi here is just complex conjugation:

$$\sqrt{2}\psi \equiv \psi_1 + i\psi_2, \sqrt{2}\bar{\psi} = \psi_1 - i\psi_2,$$

both $\psi_{1,2}$ are left-movers.

(b) Now we will discuss $\mathcal{N} = (2,2)$ supersymmetry. This means that we have a complex grassmann superspace coordinate of both chiralities $\alpha = \pm\frac{1}{2}$: $\theta_+, \bar{\theta}_+, \theta_-, \bar{\theta}_-$. And therefore we have four superspace derivatives:

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\rho^a_{\alpha\beta}\bar{\theta}^\beta\partial_a, \qquad \bar{D}_\alpha = -\frac{\partial}{\partial\bar{\theta}^\alpha} - i\rho^a_{\alpha\beta}\theta^\beta\partial_a.$$

A *chiral* superfield is one which is killed by half the supercharges:

$$\bar{D}_\pm\Phi = 0.$$

Such a field can be expanded as ($\alpha = \pm$)

$$\Phi(x, \theta, \bar{\theta}) = \phi(y) + \sqrt{2}\theta_\alpha\psi^\alpha(y) + \theta^\alpha\theta_\alpha F$$

where

$$y^a = x^a + i\theta^\alpha\sigma^a_{\alpha\beta}\bar{\theta}^\beta$$

and the $\pm$ indices are raised and lowered with $\epsilon^{\alpha\beta}$.

Show that the action

$$S_{\text{canonical}} = \int d^2z \int d^2\theta_+ d^2\theta_- \ \bar{\Phi}\Phi \quad .$$

gives a canonical kinetic term for X and its superpartner, but no term with derivatives of the *auxiliary field* F. [3]

Using the basis of σs from the phases paper, the important bits of $\bar{\Phi}\Phi$ are

$$\bar{\Phi}\Phi|_{\theta^4} = -4\bar{\phi}\partial_+\partial_-\phi - 8(\partial_+\partial_-\bar{\phi})\phi + 4\partial_-\bar{\phi}\partial_+\phi + 4\partial_+\bar{\phi}\partial_-\phi$$

$$+4F^2 + 2\left(-2i\psi_-\partial_+\bar{\psi}_- - 2i\psi_+\partial_-\bar{\psi}_+ - 2i\bar{\psi}_-\partial_+\psi_- - 2i\bar{\psi}_+\partial_-\psi_+\right)$$

A more general kinetic term comes from a *Kähler potential* K,

$$S_K \int d^2z \int d^2\theta_+ d^2\theta_- \; K(\bar{\Phi}, \Phi) \quad ,$$

where before we made the special choice $K = \bar{\Phi}\Phi$. What are the bosonic terms coming from this superspace integral?

We can just taylor expand K in its argument and use the previous result:

$$S_K = (\partial_{\bar{\Phi}}\partial_{\Phi}K)\, \partial_a\bar{\Phi}\partial^a\bar{\Phi} + fermions.$$

For more than one chiral field, we get

$$S_K = K_{ij}\partial_a\bar{\Phi}^i\partial^a\bar{\Phi}^j$$

where

$$K_{ij} \equiv \partial_{\bar{\Phi}^i}\partial_{\Phi^j}K$$

is the kahler metric; K is called the kahler potential. Notice that the transformation $K \to K + f + \bar{g}$ where f is a function only of chiral superfields, and g only of antichiral fields, does not change the form of the action. Please see the phases paper if you need the coefficients of the fermion terms.

[3]It will be useful to note that the complex conjugate field $\bar{\Phi}$ is an antichiral multiplet satisfying $\bar{D}_\alpha\bar{\Phi} = 0$, and can be expanded as

$$\bar{\Phi} = \bar{\phi}(\bar{y}) + \sqrt{2}\bar{\theta}_\alpha\bar{\psi}^\alpha(\bar{y}) + \bar{\theta}_\alpha\bar{\theta}^\alpha F$$

where $\bar{y}^a = x^a - i\theta^\alpha\sigma^a_{\alpha\beta}\bar{\theta}^\beta$.

Now consider a *superpotential* term, which can be written as an integral over only half of the superspace:

$$S_W = \int d^2z \int d\theta_+ d\theta_- W(\Phi) + \text{h.c.}.$$

Show that this term is supersymmetric if W depends only on chiral superfields in a holomorphic way, $\frac{\partial}{\partial \bar{\Phi}} W = 0$.

The supersymmetry variation of this term is

$$\left(\bar{\epsilon} D + \epsilon \bar{D}\right) \int d\theta_+ d\theta_- W.$$

We can rewrite

$$\int d\theta_+ d\theta_- = \frac{1}{2}[D_+, D_-].$$

The holomorphic D's die using $D^2 = 0$. The antiholomorphic D's because $\bar{D}W = 0$, by the chiral superfield condition.

With the action

$$S = S_K + S_W$$

integrate out the auxiliary fields $F, \bar{F}$ to find the form of the bosonic potential for ϕ. [4] Describe the supersymmetric ground states of this system.

The bosonic bits of this action are of the form ($W_i \equiv \partial_{\Phi^i} W$)

$$K_{i\bar{j}} \left(\partial\Phi^i \partial\bar{\Phi}^{\bar{j}} + F^i \bar{F}^{\bar{j}} + ferms \right) + W_i \bar{F}^i + \bar{W}_i F^i + W_{ij} \bar{\psi}^i \psi^j + h.c.$$

The equation of motion for the auxiliary field $\bar{F}^j$ is

$$K_{ij} F^i + W_i = 0;$$

which we can solve in terms of the inverse Kahler metric $K^{ij} K_{jk} = \delta^i_k$. Plugging back in we get

$$V(\Phi, \bar{\Phi}) = K^{ij} W_i \bar{W}_j.$$

This has $V = 0$ minima where $W_i = 0$, *i.e.* at critical points of the superpotential.

[4] It is often useful to label a superfield by its lowest component.

4. **Supersymmetric ghosts.** Consider a (chiral) $\mathcal{N}=2$ multiplet of ghosts:

$$B = \beta + \theta b, \quad C = c + \theta\gamma \quad ;$$

here θ is a coordinate on 2d $\mathcal{N}=2$ superspace, b, c are ordinary Grassmann bc ghosts, with scaling weight $\lambda, 1-\lambda$. β, γ are *commuting* ghosts with weights $\lambda - \frac{1}{2}$ and $\frac{1}{2} - \lambda$ respectively.

Write the action

$$S_{BC} = \int d^2z \int d^2\theta \; B\bar{D}C$$

in components; here $\bar{D} = \frac{\partial}{\partial\theta} + \bar{\theta}\bar{\partial}_{\bar{z}}$ is the same superspace covariant derivative from before.

Find an expression for the supercurrent in this theory.

In components the action is

$$S_{BC} = \int d^2z d^2\theta \, (\beta + \theta b) \left(\bar{\theta}\bar{\partial}c + \bar{\theta}\theta\bar{\partial}\gamma\right) = \int d^2z \left(\beta\bar{\partial}\gamma + b\bar{\partial}c\right) .$$