MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics
String Theory (8.821) – Prof. J. McGreevy – Fall 2007

Solution Set 3
2d CFT

Due: Thursday, October 11, 2007 at 11:00 AM, in the 8.821 lockbox.
Reading: Ginsparg CFT notes §1,2,3,6. Try §4. Polchsinski, Chapter 2. GSW, Chapter 3 (note that the CFT is hidden in §3.2, entitled 'BRST quantization').

1. **Warmup.** Compute the scaling dimension of the operator $\mathcal{O}_k \equiv : e^{ikX} :$ in the free boson theory whose stress tensor is $T(z) = -\frac{1}{\alpha'} : \partial X \partial X(z) :$, using the $T\mathcal{O}_k$ OPE.

$$T(z)\mathcal{O}_k(0) = -\frac{1}{\alpha'} : \partial X \partial X(z) :: e^{ikX}(0) :$$

$$= -\frac{1}{\alpha'} \left((ik)^2 \left(\partial_z \langle X(z)X(0)\rangle\right)^2 + ik \cdot \partial X(z) \partial_z \langle X(z) X(0)\rangle \right) : e^{ikX}(0) :$$

When T has the given normalization (as in Polchinski), the propagator is

$$\langle X(z)X(0)\rangle = -\frac{\alpha'}{2} \ln z + \text{antiholomorphic}$$

$$\partial_z \langle X(z)X(0)\rangle = -\frac{\alpha'}{2z}$$

Which gives

$$T(z)\mathcal{O}_k(0) = \left(\frac{\alpha' k^2}{4} \frac{1}{z^2} + \frac{1}{z}\partial \right) : e^{ikX}(0) : +\text{regular}.$$

2. **Linear dilaton CFT.**

The *linear dilaton theory* is a 2d CFT made from a free boson in the presence of a 'background charge.' This means that the boson X has some funny coupling to the worldsheet gravity, which can be described by a linear dilaton term in the action $S_\Phi = \int d^2\sigma \Phi(X) R^{(2)}$, $\Phi(X) = QX$, Q is a constant. On a flat worldsheet, the quantization of X proceeds as before. In that case, the only difference from the ordinary free boson is that the stress tensor (which is sensitive to how the theory is coupled to gravity) has the form

$$T_Q = -\frac{1}{\alpha'} : \partial X \partial X : + V \partial^2 X$$

(where $V \sim Q$).

[Optional: relate V to Q.]

(a) Verify that T_Q has the right OPE with itself to be the stress tensor for a CFT. Compute the Virasoro central charge for the linear dilaton theory.

The best way to relate V and Q is to extract the stress tensor from the coupling to gravity: $T_{ab} = \frac{2}{\sqrt{g}} \frac{\delta S}{\delta g^{ab}}$. To derive it from Noether's method one needs to realize that X will *shift* under conformal transformations, as you can check by studying $\delta_\xi X = -\oint \frac{dz}{2\pi i} \xi(z) T_Q(z) X$.

$$T_Q(z) T_Q(w) = \left(-\frac{1}{\alpha'} : \partial X \partial X(z) : + V \partial^2 X(z) \right) \left(-\frac{1}{\alpha'} : \partial X \partial X(w) : + V \partial^2 X(w) \right)$$

The propagator for X is as before (the EOM is unchanged by the linear dilaton term). The contractions between the ∂X^2 terms give the same answer as we got for ordinary free bosons:

$$\left(-\frac{1}{\alpha'} \right)^2 : \partial X \partial X(z) :: \partial X \partial X(w) :=$$

$$\frac{2}{\alpha'^2} \left(\left(\partial_z \partial_w \left(\frac{-\alpha'}{2} \right) \ln(z-w) \right)^2 + 2 : \partial X(z) \partial X(w) : \partial_z \partial_w \left(\frac{-\alpha'}{2} \right) \ln(z-w) \right)$$

$$= \frac{1/2}{(z-w)^4} + \frac{2}{(z-w)^2} : -\frac{1}{\alpha'} \partial X(w)^2 : + \frac{\partial_w}{z-w} : -\frac{1}{\alpha'} \partial X(w)^2 :$$

One cross term is:

$$-\frac{1}{\alpha'} : \partial X \partial X(z) : V \partial^2 X(w) = -\mathbf{2} \frac{V}{\alpha'} \partial_z \partial_w^2 \langle X(z) X(w) \rangle \partial X(w)$$

$$= + \frac{2}{(z-w)^3} \left(\partial X(w) + (z-w) \partial^2 X(w) + \frac{1}{2} (z-w)^2 \partial^3 X(w) + \ldots \right)$$

$$= -\frac{2V}{(z-w)^3}\partial X(w) + \frac{2V}{(z-w)^2}\partial^2 X(w) + \frac{V}{(z-w)}\partial^3 X(w) + \text{regular.}$$

The other cross term is:

$$-\frac{1}{\alpha'}V\partial^2 X(z) : \partial X \partial X(w) := -\mathbf{2}\frac{V}{\alpha'}\partial_z^2 \partial_w \langle X(w)X(z)\rangle \partial X(z)$$

$$= +\frac{2V}{(w-z)^3}\partial X(w)$$

The V^2 term only contributes to c:

$$V\partial^2 X(z) V\partial^2 X(w) = V^2 \partial_z^2 \partial_w^2 \frac{-\alpha'}{2}\ln(z-w) = +\frac{\alpha' V^2}{2}\frac{6}{(z-w)^4}$$

Adding these up gives:

$$TT = \frac{1}{(z-w)^4}\left(\frac{1+6\alpha' V^2}{2}\right) + \frac{1}{(z-w)^3}\frac{-V}{2\alpha'}\left(\frac{2}{(z-w)^3}\partial X(w) - \frac{2}{(z-w)^3}\partial X(w)\right)$$

$$+\frac{2}{(z-w)^2}\left(: -\frac{1}{\alpha'}\partial X(w)^2 : + V\partial^2 X(w)\right) + \frac{1}{z-w}\partial_w\left(: -\frac{1}{\alpha'}\partial X(w)^2 : + V\partial^2 X(w)\right)$$

so the dangerous $(z-w)^{-3}$ terms die a grisly death leaving exactly the usual TT OPE with central charge

$$c(V) = 1 + 6\alpha' V^2.$$

(b) Compute the scaling dimension of the operator $: e^{ikX} :$ in the linear dilaton theory.

Extra stimulation: Can you interpret the result of (b) in terms of a target space effective action?

The terms where the $(\partial X)^2$ hits the vertex op is just like in the first problem. The rest is

$$V\partial^2 X(z) : e^{ikX}(0) := ik\cdot V\partial_z^2\left(-\frac{\alpha'}{2}\ln z\right) : e^{ikX}(0) := \frac{1}{z^2}\left(-\frac{\alpha'}{2}ik\cdot V\right) : e^{ikX}(0) :$$

This corrects the conformal dimension to

$$h_V(k) = \frac{\alpha' k^2}{4} - \frac{\alpha'}{2}ik\cdot V.$$

Recalling that physical states satisfy $h = 1$, the mass shell condition for the tachyon vertex in a linear dilaton theory is of the form

$$0 = h_V(k) - 1 = \frac{\alpha' k^2}{4} - \frac{\alpha'}{2} ik \cdot V - 1.$$

This should be the momentum-space wave equation for the target space theory of the tachyon. The tree level terms quadratic in the tachyon are

$$S_2[T, \Phi] = - \int d^D x \ e^{-2\Phi(x)} \left(\partial_\mu T \partial^\mu T + m_T^2 T^2 \right),$$

where the $e^{2\Phi}$ prefactor comes from the fact that the euler character of the sphere is 2. If the dilaton has a profile $\Phi(x) = Qx$, the equation of motion for T is

$$0 = \frac{\delta S}{\delta T(y)} = -2 \int d^D x \ e^{-2Qx} \left(\partial_\mu \delta^D(x-y) \partial^\mu T + m_T^2 T \delta^d(x-y) \right).$$

When we integrate by parts to get the derivative off the delta function, it hits the linear dilaton:

$$0 = -2Q\partial_x T + \partial^2 T - m_T^2 T$$

which in momentum space gives the equation we got from $h_V(k)$. Apparently $Q = V$.

3. **The stress tensor is not a conformal primary if $c \neq 0$.** [1]

   (a) For any 2d CFT, use the general form of the TT OPE to show that the transformation of T under an infinitesimal conformal transformation $z \mapsto z + \xi(z)$ is

$$-\delta_\xi T(w) = (\xi \partial + 2\partial \xi) T(w) + \frac{c}{12} \partial^3 \xi. \qquad (1)$$

   The variation of T under a conformal transformation parametrized by a holomorphic vector field $\xi(z)$ is [2]

[1] I got this problem from Robbert Dijkgraaf.

[2] Note that the expression in the first version of this pset differs by a sign from the one in Polchinski; the sign comes from $\delta_\xi T(0) = i[L[\xi], T(0)]$ and $L[\xi] = i \oint \xi T$. It could be absorbed by $\xi \to -\xi$, but gets confusing when we consider the finite transformation next (*i.e.* $-\xi$ generates the *inverse* transformation).

$$
\begin{aligned}
-\delta_\xi T(0) &= \oint_{C_0} \frac{dz}{2\pi i} \xi(z) T(z) T(0) \\
&= \oint_{C_0} \frac{dz}{2\pi i} \xi(z) \left(\frac{c/2}{z^4} + \frac{2T}{z^2} + \frac{\partial T}{z} + \text{reg.} \right) \\
&= \frac{c \partial^3 \xi(0)}{2 \cdot 3!} + 2\partial\xi(0) T(0) + \xi(0) \partial T(0).
\end{aligned}
$$

(b) Consider the *finite* conformal transformation $z \mapsto f(z)$. Show that (1) is the infinitesimal version of the transformation law

$$
T_{zz}(z) = (\partial f)^2 T_{ff}(f(z)) + \frac{c}{12}\{f, z\} \qquad (\textit{Elephant})
$$

where

$$
\{f, z\} \equiv \frac{\partial f \partial^3 f - \frac{3}{2}(\partial^2 f)^2}{(\partial f)^2}
$$

is called a *Schwarzian derivative.*

[Optional: verify that this extra term does the right thing when composing two maps $z \to f(z) \to g(f(z))$.]

Infinitesimally, $f(z) = z + \xi, \partial f = 1 + \partial \xi, \partial^2 f = \partial^2 \xi ...$**, and the Schwarzian deriv is**

$$
\{z + \xi, z\} = \frac{(1 + \partial\xi)\partial^3\xi - 3(\partial^2\xi)^2}{(1 + \partial\xi)^2} = \partial^3\xi + \mathcal{O}(\xi^2).
$$

So $(\textit{Elephant})$ **becomes**

$$
T_{zz}(z) = (1+\partial\xi)^2 T_{ff}(z+\xi) + \frac{c}{12}\{z+\xi, z\} = T_{ff}(z) + 2\partial\xi T(z) + \xi\partial T + \frac{c\partial^3\xi}{12} + \mathcal{O}(\xi^2)
$$

which gives

$$
\delta_\xi T = (T_{ff} - T_{zz})_{f=z+\xi} = -\left(2\partial\xi T(z) + \xi\partial T + \frac{c\partial^3\xi}{12} \right) + \mathcal{O}(\xi^2)
$$

which agrees with our expression (1) **from the part (a).**

To show that $(\textit{Elephant})$ **is the correct exponentiation of the infinitesimal transformation we also need to show that it composes correctly under successive transformations. This means**

$$
T_{zz}(z) \mapsto (\partial f)^2 T_{ff}(f(z)) + \frac{c}{12}\{f, z\} = (\partial_z f)^2 \left((\partial_f g)^2 T_{gg}(g(f(z))) + \frac{c}{12}\{g, f\} \right) + \frac{c}{12}\{f, z\}
$$

This should give the same result as

$$T_{zz}(z) \mapsto (\partial g)^2 T_{gg}\left(g(z)\right) + \frac{c}{12}\{g, z\}$$

which requires that

$$\{g, z\} - \{f, z\} = (\partial_z f)^{-2}\{g, f\}$$

which is the case since the BHS can be seen (after a bit of cooking) to equal

$$BHS = \frac{2\left(\partial^3 g(\partial f)^2 \partial g - \partial^3 f(\partial g)^2 \partial f\right) - 3\left((\partial^2 g)^2(\partial f)^2 - (\partial^2 f)^2(\partial g)^2\right)}{2(\partial_z g)^2(\partial_z f)^{-4}}.$$

(c) Given that the conformal map from the cylinder to the plane is $z = e^{-iw}$, show that (b) means that

$$T_{\text{cyl}}(w)(dw)^2 = \left(T_{\text{plane}}(z) + \frac{c}{24}\right)(dz)^2.$$

Use this relation to show that the Hamiltonian on the cylinder

$$H = \int \frac{d\sigma}{2\pi} T_{\tau\tau}$$

is

$$H = L_0 + \tilde{L}_0 - \frac{c + \bar{c}}{24}.$$

Comment: After all this complication, the result has a very simple physical interpretation: when putting a CFT on a cylinder, the scale invariance is spontaneously broken by the fact that the cylinder has a *radius*, *i.e.* the cylinder introduces a (worldsheet) length scale into the problem. The term in the energy extensive in the radius of the cylinder (and proportional to c) is actually experimentally observable.

Given that $\partial_z w = \frac{i}{z}$, the Schwarzian is

$$\{w, z\} = \frac{(2i/z^3)(i/z) - 3/2(-i/z^2)^2}{(i/z)^2} = z^{-2}(2 - 3/2) = \frac{1}{2z^2} = -\frac{1}{2}(\partial_z w)^2.$$

Therefore, our previous expression gives

$$T_{zz}(z) = T_{ww}(w)(\partial_z w)^2 + \frac{c}{12}\{w, z\} = \left(T_{ww}(w) - \frac{c}{24}\right)(\partial_z w)^2.$$

Multiplying by $(dz)^2$, this gives the desired relation. The relation for the hamiltonian just requires remembering that $T_{\tau\tau} = T_{zz} + T_{\bar{z}\bar{z}}$.

4. **Constraints from Unitarity.** Show that in a unitary CFT, $c > 0$, and $h \geq 0$ for all primaries. Hint: consider $\langle\phi|[L_n, L_{-n}]|\phi\rangle$.

   Consider a primary state $|\phi\rangle$ of weight h in a unitary CFT. In such a CFT (unlike the CFT with X^0 or FP ghosts), for any n,

   $$0 \leq ||L_{-n}|\phi\rangle||^2 = \langle\phi|L_n L_{-n}|\phi\rangle = \langle\phi|[L_n, L_{-n}]|\phi\rangle \qquad (Roger)$$

   where in the last equality we used the fact that ϕ is primary and hence annihilated by L_n, $n > 0$. Consider first $n = 1$, and use Vir (actually just $sl(2)$) to say:

   $$0 \leq \langle\phi|[L_1, L_{-1}]|\phi\rangle = \langle\phi|(2L_0)|\phi\rangle 2h|||\phi\rangle||^2$$

   which is true iff $h \geq 0$. The inequality is only saturated ($h = 1$) if $L_1|\phi\rangle = 0$, which means that the state is killed by the whole $sl(2)$, which identifies it as the conformal vacuum, *i.e.* the image of the identity operator under the state-operator correspondence. More precisely, it means that the operator ϕ satisfies $\partial\phi(z) = 0$ (remember $L_1 \sim \partial$), *i.e.* it's constant, which means that it must be proportional to the identity operator (see p. 48 of Polchinski for the argument).

   To see that $c > 0$, let's use Vir on $(Roger)$ to get

   $$0 \leq \langle\phi|[L_n, L_{-n}]|\phi\rangle = 2nh + c(n^3 - n) = n(2h - c) + n^3 c.$$

   For small n, this could be true with negative c if h were big enough, but for large enough n, the second, n^3, term kicks the pants off the first term and had better have a positive coefficient.

5. $SU(2)_1$ **current algebra from a circle.**

   Consider the closed bosonic string compactified on a circle of radius $R = \sqrt{\alpha'}$. In lecture 4 all kinds of ridiculous claims were made about this theory. Here we will study the CFT describing the strings on this circle and verify that there is in fact an $SU(2)_L \times SU(2)_R$ gauge symmetry involving winding modes. We'll focus on the holomorphic (L) part; the antiholomorphic part will be identical.

   Label the circle coordinate $X^{25} \equiv X \sim X + 2\pi R$. Define

   $$J^{\pm}(z) \equiv :e^{\pm 2iX(z)/\sqrt{\alpha'}}:, \qquad J^3 \equiv i\sqrt{\frac{2}{\alpha'}}\partial X(z).$$

   (a) Show that $J^3, J^{\pm}$ are single-valued at the 'self-dual radius' $R = \sqrt{\alpha'}$.

(b) At the self-dual radius, do $J^\pm, J^3$ have the right conformal dimension to create physical string states which are massless?

Clearly the J^3 current is just the ordinary momentum current which we've previously checked has dimension $(1,0)$. To save typing, let's define $k \equiv \frac{2i}{\sqrt{\alpha'}}$. The $TJ^\pm$ OPE is

$$\frac{T(z)J^\pm(0) \sim -\frac{1}{\alpha'}\left(k\frac{-\alpha'}{2z}\right)^2 J^\pm(0) + \mathcal{O}(1/z) = h}{z^2 J^\pm + \mathcal{O}(1/z)}$$

with

$$h = -\frac{k^2\alpha'}{4} = 1.$$

(c) Defining $J^\pm \equiv \frac{1}{\sqrt{2}}(J^1 \pm iJ^2)$ show that the operator product algebra of these currents is

$$J^a(z)J^b(0) \sim \frac{\delta^{ab}}{z^2} + i\sqrt{2}\epsilon^{abc}\frac{J^c(0)}{z} + ...$$

$$J^1 = \frac{1}{\sqrt{2}}(J^+ + J^-), J^1 = -\frac{i}{\sqrt{2}}(J^+ - J^-).$$

$$J^3(z)J^1(0) = i\sqrt{\frac{2}{\alpha'}}\partial X(z)\frac{1}{\sqrt{2}}\left(e^{+kX} + e^{-kX}\right)$$

$$\sim \frac{i}{\sqrt{\alpha'}}\partial_z(-\frac{\alpha'}{2}\ln z)k : \left(e^{+kX} - e^{-kX}\right)(0) := -\frac{1}{z}i\frac{\sqrt{\alpha'}}{2}\frac{2i}{\sqrt{\alpha'}} : \left(e^{+kX} - e^{-kX}\right)(0) :$$

$$= \frac{1}{z} : \left(e^{+kX} - e^{-kX}\right)(0) := i\sqrt{2}J^1(0).$$

Let's check for the kronecker delta term:

$$J^1(z)J^1(0) = \left(\frac{1}{\sqrt{2}}\right)^2 \left(e^{+kX} + e^{-kX}\right)\left(e^{+kX} + e^{-kX}\right)$$

$$\sim \frac{1}{2}\Big(z^{\frac{-\alpha'}{2}k^2} : e^{kX(z)+kX(0)} : +z^{\frac{-\alpha'}{2}(-k)^2} : e^{-kX(z)-kX(0)} : \\ +z^{-\frac{-\alpha'}{2}k(-k)}\left(: e^{kX(z)-kX(0)} : + : e^{kX(0)-kX(z)} :\right)\Big) \quad (1)$$

$$\sim \frac{1}{2}\left(\frac{1}{z^2}\left(2+\mathcal{O}(z^2)\right)+\mathcal{O}(z^2)\right)$$

An interesting one is

$$J^1(z)J^2(0) = \left(\frac{1}{\sqrt{2}}\right)^2 (-i)\left(e^{+kX}+e^{-kX}\right)\left(e^{+kX}-e^{-kX}\right)$$

$$= -\frac{i}{2}\left[\frac{1}{z^2}\left(:e^{-kX(z)+kX(0)}: - :e^{-kX(0)+kX(z)}:\right)+\mathcal{O}(z^2)\right)$$

$$= -\frac{i}{2}\left[\frac{1}{z^2}\left(:e^{-k(z\partial X(0)+...)}: - :e^{+k(z\partial X(z)+...)}:\right)\right] = -\frac{i}{2}\frac{1}{z}\left(-2k\partial X(0)\right)+...$$

$$\sim \frac{i\sqrt{2}}{z}J^3(0).$$

The others are similar.

(d) [Bonus tedium] Defining modes as usual for a dimension 1 operator,

$$J^a(z) = \sum_{n\in\mathbb{Z}} J^a_n z^{-n-1}$$

show that

$$[J^a_m, J^b_n] = i\sqrt{2}\epsilon^{abc}J^c_{m+n} + mk\delta^{ab}\delta_{m+n}$$

with $k=1$, which is an algebra called Affine $SU(2)$ at level $k=1$. Note that the $m=0$ modes satisfy the ordinary $SU(2)$ lie algebra.

$$[J^a_m, J^b_n] = \oint_{C_0}\frac{dw}{2\pi i}\oint_{C_w}\frac{dz}{2\pi i}J^a(z)J^b(w) = \oint_{C_0}\frac{dw}{2\pi i}\oint_{C_w}\frac{dz}{2\pi i}\left(\frac{\delta^{ab}}{(z-w)^2}+i\sqrt{2}\epsilon^{abc}\frac{J^c(w)}{z-w}\right)$$

$$= \oint_{C_0}\frac{dw}{2\pi i}w^m\left(\partial_z z^n|_{z=w}\delta^{ab}+i\sqrt{2}\epsilon^{abc}z^n|_{z=w}J^c(w)\right)$$

$$= \oint_{C_0}\frac{dw}{2\pi i}\left(nw^{m+n-1}\delta^{ab}+i\sqrt{2}\epsilon^{abc}J^c(w)w^{m+n}\right)$$

which gives the desired mode algebra.

(e) Think about how the results of (a)-(d) verify the claim that the spectrum of the compactified theory at this special radius really has non-abelian gauge symmetry, with the extra gauge bosons made from wound strings. Specifically, construct the physical vertex operators for the three $SU(2)_L$ gauge bosons by tensoring J^a with some right-moving operator (remember to level-match) and some factor that allows the resulting state to have (null) momentum in the noncompact dimensions. Remember that an operator $e^{ik_L X_L} e^{ik_R X_R}$ creates a string mode with nonzero winding if $k_L \neq k_R$.

A good set of closed-string vertex operators for the gauge bosons are

$$V^{a\mu}(k) \equiv J^a(z)\bar{\partial}X^\mu(\bar{z}) : e^{i\sum_{\mu=0}^{24} k_\mu X^\mu} : \qquad .$$

These create massless vectors because they have a vector index, and the condition that $(h, \bar{h}) = (1, 1)$ is $k_\mu k^\mu = 0$. It is a winding mode because $k_L^{25} \neq k_R^{25}$.