MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics
String Theory (8.821) – Prof. J. McGreevy – Fall 2007

Solution Set 1

Classical Worldsheet Dynamics

Reading: GSW §2.1, Polchinski §1.2-1.4. Try §3.2-3.3.

Due: Friday, September 21, 2007 at 3 PM, in the 8.821 lockbox.

1. **Weyl symmetry.**[1] A (global) Weyl transformation acts on a field theory coupled to gravity by

$$F(x) \mapsto \lambda^{-W(F)} F(x)$$

where F is a field, and $W(F)$ is a number called the *Weyl weight* of F. W is normalized so that $W(g_{\mu\nu}) = 2$.

(a) Find $W(\varphi)$ so that

$$S = \int d^D x \sqrt{g} \frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi$$

is Weyl invariant.

$$\varphi(x) \mapsto \lambda^{-W(\varphi)} \varphi(x)$$

$$g_{\mu\nu} \mapsto \lambda^{-2} g_{\mu\nu}$$

$$g^{\mu\nu} \mapsto \lambda^{+2} g^{\mu\nu}$$

$$\det g \mapsto \lambda^{-2D} \det g$$

$$e \equiv \sqrt{g} \mapsto \lambda^{-D} e$$

$$S \mapsto S \lambda^{-D+2-2W(\varphi)} \stackrel{!}{=} S$$

$$W(\varphi) = \frac{2-D}{2}$$

[1] Problems 1-4 are due to Marty Halpern.

(b) Consider coordinate transformations of the form $x \mapsto x' = \lambda^{-1}x$, which can be called Einstein-scale transformations. How do φ and $g_{\mu\nu}$ transform?

$$\frac{\partial x'^{\ \mu}}{\partial x^{\nu}} = \lambda^{-1}\delta^{\mu}_{\nu}$$
$$\varphi(x) \mapsto \varphi(x') = \varphi(\lambda^{-1}x)$$
$$g_{\mu\nu} \mapsto \frac{\partial x^{\rho}}{\partial x'^{\mu}}\frac{\partial x^{\sigma}}{\partial x'^{\nu}} g_{\rho\sigma}(\lambda^{-1}x) = \lambda^{2} g_{\rho\sigma}(\lambda^{-1}x).$$

(c) Write down the transformation of φ and $g_{\mu\nu}$ under simultaneous Weyl and Einstein-scale transformations. How does the result for φ relate to the engineering dimension you would assign φ in order to make S dimensionless?

$$\varphi(x) \mapsto \lambda^{-W(\varphi)}\varphi(\lambda x), \quad g_{\mu\nu} \mapsto g_{\rho\sigma}(\lambda x)$$

Altogether, this is just an ordinary scale transformation, where φ has its engineering mass dimension.

2. **Local Weyl Invariance.** Consider a gravity plus matter system

$$S = S_{grav}[g_{\mu\nu}] + S_{matter}[F, g_{\mu\nu}]$$

where S_{matter} is invariant under the *local* Weyl transformation

$$F(x) \mapsto \lambda(x)^{-W[F]}F(x), \quad g_{\mu\nu}(x) \mapsto \lambda^{-2}(x) g_{\mu\nu}(x).$$

Using the equations of motion, show that the matter stress tensor

$$T^{\mu\nu} = -\frac{2}{\sqrt{g}}\frac{\delta S_{matter}}{\delta g_{\mu\nu}}$$

is traceless:

$$T^{\mu\nu}g_{\mu\nu} = 0.$$

The infinitesimal form of the Weyl transformation on the metric, with $\lambda = e^{\alpha}$ is

$$\delta_{weyl} g_{\mu\nu}(x) = -2\alpha g_{\mu\nu}(x).$$

Now consider a general variation of S_{matter}:

$$\delta S_{matter} = \int d^D x \left(\frac{\delta S_{matter}}{\delta g_{\mu\nu}(x)}\delta g_{\mu\nu}(x) + \frac{\delta S_{matter}}{\delta F(x)}\delta F(x) \right)$$

around a classical solution for the matter fields, this is

$$\int d^D x \left(\frac{\delta S_{matter}}{\delta g_{\mu\nu}(x)} \delta g_{\mu\nu}(x) \right) = \int d^D x \left(-\frac{\det g}{2} T^{\mu\nu} \delta g_{\mu\nu} \right).$$

Now in the particular case of a weyl transformation, since were assuming S_{matter} is weyl invariant,

$$0 = \delta_{weyl} S_{matter} = \int d^D x \alpha(x) \left(\sqrt{g} T^{\mu\nu} g_{\mu\nu} \right).$$

Since $\alpha(x)$ is arbitrary, this implies

$$T^{\mu\nu} g_{\mu\nu} = 0.$$

Note that any (globally) Weyl-invariant matter action gives a traceless stress tensor in the flat space limit ($g_{\mu\nu} \to \eta_{\mu\nu}$).

3. **Symmetries of the Polyakov action.** Recall the Polyakov action

$$S_P = -\frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-\gamma} \gamma^{ab} \partial_a X^\mu \partial_b X_\mu.$$

Think of this as the action for a system of matter (X) coupled to 2d gravity (γ_{ab}).

(a) Show that the Polyakov action is locally Weyl-invariant.

In the first problem we learned that the weyl-weight for X in $D = 2$ is $\frac{2-D}{2} = 0$, so the local weyl transformation acts as

$$g_{ab}(\sigma) \mapsto \lambda(\sigma)^{-2} g_{ab}(\sigma), \quad X \mapsto X$$

From the transformation of the metric, we have also

$$\sqrt{g} \mapsto \lambda^{-2} \sqrt{g}$$

$$g^{ab} \mapsto \lambda^2 g^{ab}$$

$$\sqrt{g} g^{ab} \mapsto \sqrt{g} g^{ab}$$

so S_P is locally Weyl invariant. This is something special about strings, compared to particles or membranes or...

(b) Obtain an expression for the stress tensor

$$T_{ab} \equiv -\frac{2}{\sqrt{\gamma}} \frac{\delta S_P}{\delta \gamma^{ab}}$$

and check that it is traceless. [You may want to use the relation

$$\delta(\det A) = \det A \ \mathrm{tr}\ (A^{-1} \delta A),$$

where A is any nonsingular matrix, and δA is any small deformation.]

For a scalar field ϕ coupled to a background metric by the lagrangian density

$$\mathcal{L} \equiv D^a \phi D_b \phi - V(\phi)$$

with action

$$S = \int d^D x \sqrt{g} \mathcal{L}$$

the stress tensor is

$$T^{ab} = -\frac{2}{\sqrt{g}} \frac{\delta S}{\delta g_{ab}} = D^a \phi D^b \phi - g^{ab} \mathcal{L}.$$

Making the replacement

$$\phi = X^\mu, D = 2, V = 0, S_P = -\frac{1}{4\pi\alpha'} \sum_\mu S[X^\mu],$$

we get

$$T^{ab} = -\frac{1}{4\pi\alpha'} \left(D^a X^\mu D^b X_\mu - \frac{1}{2} \gamma^{ab} \partial_c X^\mu \partial_d X_\mu \right), \qquad \text{where } D^a X^\mu \equiv \gamma^{ab} \partial_b X^\mu.$$

Using $\gamma^{ab} \gamma_{ab} = D = 2$, the trace is

$$\Theta \equiv T^{ab} \gamma_{ab} = -\frac{1}{4\pi\alpha'} \left(\gamma^{ab} \partial_a X^\mu \partial_b X_\mu - \frac{2}{2} \gamma^{cd} \partial_c X^\mu \partial_d X_\mu \right) = 0.$$

This was guaranteed by local Weyl invariance via the previous problem.

(c) Use the equation of motion for γ_{ab} to show that when evaluated on a solution

$$S_P[\gamma|_{EOM}, X] = S_{Nambu-Goto}[X].$$

The equations of motion for the worldsheet metric γ_{ab} are

$$0 = T^{ab}.$$

From part (a), (and non-degeneracy of γ_{ab}) this gives

$$0 = T_{ab} \propto \partial_a X^\mu \partial_b X_\mu - \frac{1}{2}\gamma_{ab}\gamma^{cd}\partial_c X^\mu \partial_d X_\mu$$

which says

$$\partial_a X^\mu \partial_b X_\mu = \gamma_{ab}\left(\frac{1}{2}\gamma^{cd}\partial_c X^\mu \partial_d X_\mu\right) \qquad (Wilma)$$

The LHS of this last equation (which I will call $(Wilma)$) is the induced metric

$$\Gamma_{ab} \equiv \partial_a X^\mu \partial_b X_\mu.$$

Taking the determinant of the both hand side of $(Wilma)$ we learn that

$$\Gamma \equiv \det_{ab} \Gamma_{ab} = \det \gamma_{ab}\left(\frac{1}{2}\gamma^{cd}\partial_c X^\mu \partial_d X_\mu\right)^2.$$

$$\implies \sqrt{|\Gamma|} = \sqrt{|\gamma|}\frac{1}{2}\gamma^{cd}\partial_c X^\mu \partial_d X_\mu$$

which is exactly the Polyakov lagrange density. So we find

$$S_P[\gamma|_{EOM}, X] = -\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{|\gamma|}\gamma^{cd}\partial_c X^\mu \partial_d X_\mu|_{EOM} = -\frac{1}{2\pi\alpha'}\int d^2\sigma\sqrt{|\Gamma|} = S_{NG}.$$

4. **Virasoro algebra (no central extension yet).** Consider the classical mechanics of D two-dimensional free bosons X^μ, which enjoy the canonical Poisson bracket

$$\{X^\mu(\sigma), P^\mu(\sigma')\} = \eta^{\mu\nu}\delta(\sigma - \sigma')$$

with their canonical momenta P^μ. Show that the objects

$$T_{\pm\pm} \equiv \pm\frac{\pi\alpha'}{2}W_\pm^2, \quad W_\pm^\mu \equiv P^\mu \pm \frac{\partial_\sigma X^\mu}{2\pi\alpha'}$$

satisfy two commuting copies of the Virasoro algebra

$$\{T_{\pm\pm}(\sigma), T_{\pm\pm}(\sigma'\} = (T_{\pm\pm}(\sigma) + T_{\pm\pm}(\sigma'))\partial_\sigma\delta(\sigma - \sigma')$$

$$\{T_{\pm\pm}(\sigma), T_{\mp\mp}(\sigma')\} = 0.$$

You may use the identity

$$f(\sigma)f(\sigma')\partial_\sigma\delta(\sigma-\sigma') = \frac{1}{2}(f^2(\sigma)+f^2(\sigma'))\partial_\sigma\delta(\sigma-\sigma'), \forall f \qquad (Wilbur).$$

[This problem may be an opportunity to apply the Tedious Problem Rule discussed in class.]

First compute

$$\left\{W^\mu_\pm(\sigma), W^\nu_\pm(\sigma')\right\} = \pm\frac{2}{2\pi\alpha'}\eta^{\mu\nu}\partial\sigma\delta(\sigma-\sigma')$$

$$\left\{W^\mu_\pm(\sigma), W^\nu_\mp(\sigma')\right\} = 0$$

This second equation tells us immediately that

$$\{T_{\pm\pm}(\sigma), T_{\mp\mp}(\sigma')\} = 0.$$

For the other one, we have to work:

$$\{T_{\pm\pm}(\sigma), T_{\pm\pm}(\sigma'\} = \left(\frac{2\pi\alpha'}{2}\right)^2\left\{W^2_\pm(\sigma), W^2_\mp(\sigma')\right\}$$

$$= \left(\frac{2\pi\alpha'}{2}\right)^2 W_\pm(\sigma)\cdot W_\pm(\sigma')\left(\pm\frac{2}{2\pi\alpha'}\right)\partial_\sigma\delta(\sigma-\sigma')$$

$$= \pm\frac{\pi\alpha'}{2}\left[W^2_\pm(\sigma)+W^2_\pm(\sigma')\right]\partial_\sigma\delta(\sigma-\sigma')$$

where the second step used the $\{AB, C\} = A\{B, C\} + \{A, C\}B$ **identity enjoyed by Poisson brackets, and the last step used the identity** $(Wilbur)$ **above. Comparing with the expression for** $T_{\pm\pm}$ **gives the other commutator.**

5. **Spinning strings in AdS.** [This problem might be hard.[2]] In this problem we are going to think about the behavior of a string propagating in 5d anti de Sitter space (AdS). Specifically, we are going to study and use some of its

[2] It is taken from a recent paper. For privacy's sake, let's call the authors Steve G., Igor K, and Alexandre P. No, that's too obvious. Uhh... let's say S. Gubser, I. Klebanov and A. Polyakov. OK, it's hep-th/0204041.

conserved charges. When these conserved quantities are large, they can be compared to their counterparts in a dual 4d gauge theory.

(a) Write down the Nambu-Goto action for a bosonic string propagating in AdS_5, whose metric (in so-called global coordinates) is

$$ds^2 = R^2(-\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho d\Omega_3^2) \equiv G_{ij}dX^i dX^j \qquad (1)$$

where $d\Omega_3^2$ denotes the line element on the unit three-sphere,

$$d\Omega_3^2 = d\theta_1^2 + \sin^2\theta_1(d\theta_2^2 + \sin^2\theta_2 d\phi^2).$$

$$S_{NG} = -\frac{1}{2\pi\alpha'}\int d^2\sigma\sqrt{-\det_{ab}\Gamma_{ab}}$$

where the induced metric Γ is

$$\Gamma_{ab} = \begin{pmatrix} \dot{X}\cdot\dot{X} & \dot{X}\cdot X' \\ X'\cdot\dot{X} & X'\cdot X' \end{pmatrix}_{ab},$$

where as usual $\dot{X} = \partial_\tau X, X' = \partial_\sigma X, A\cdot B = G_{ij}A^iB^j$. So

$$\Gamma_{\tau\tau} = R^2\left(1 + \sinh^2\rho\left(\dot{\theta}_1^2 + \sin^2\theta_1\left(\dot{\theta}_2^2 + \sin^2\theta_2\dot{\phi}^2\right)\right)\right)$$

etc..

(b) The AdS background has many isometries. We will focus on two: shifts of t (the energy) and shifts of ϕ (the spin). These isometries of the target space are symmetries of the NLSM, and therefore lead to conserved charges. Using the Noether method, write an expression for the conserved charge S which follows from the symmetry $\phi \to \phi + \epsilon$, with ϵ a constant. Write an expression for the conserved charge E which follows from the symmetry $t \to t + \delta$, with δ a constant.

The Noether method says that:

$$\delta S = \int d^2\sigma \partial_a \epsilon J^a.$$

With $\delta\varphi = \epsilon(\sigma,\tau)$, the variation of the induced metric Γ_{ab} is

$$\delta\Gamma_{ab} = R^2\sinh^2\rho\sin^2\theta_1\sin^2\theta_2\partial_a\epsilon\partial_b\varphi.$$

Using $\delta\sqrt{\Gamma} = \frac{1}{2}\sqrt{\Gamma}\Gamma^{ab}\delta\Gamma_{ab}$,

$$\delta S_{NG} = -\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{\Gamma}\Gamma^{ab}\delta\Gamma_{ab}.$$

More directly, this says that

$$J_0^{spin} = -\frac{1}{2\pi\alpha'}\frac{1}{\sqrt{\det\Gamma}}[G_{\phi\phi}\dot{\phi}(X')^2 - G_{\phi\phi}\phi'\dot{X}\cdot X']$$

So that the spin is

$$S \equiv \int_0^{2\pi} d\sigma J_0^{spin}(\sigma, \tau = const) = -\frac{1}{2\pi\alpha'}\int d\sigma\frac{1}{\sqrt{\det\Gamma}}[G_{\phi\phi}\dot{\phi}(X')^2 - G_{\phi\phi}\phi'\dot{X}\cdot X']$$

Similarly, if $\delta t = \epsilon(\sigma, \tau)$ the variation of the induced metric Γ_{ab} is

$$\delta\Gamma_{ab} = R^2\cosh^2\rho\partial_a\partial_b\epsilon.$$

$$J_0^{energy} = -\frac{1}{2\pi\alpha'}\frac{1}{\sqrt{\det\Gamma}}[G_{tt}\dot{t}(X')^2 - G_{tt}t'\dot{X}\cdot X']$$

and the energy is

$$E \equiv \int_0^{2\pi} d\sigma J_0^{energy}(\sigma, \tau = const) = ...$$

We're going to consider a spinning folded string, which spins as a rigid rod around its center, and lies on an equator of the S^3 in (1), $\theta_1 = \theta_2 = \pi/2$. The center of the string is at the center of AdS, $\rho = 0$. Go to a static gauge $t = \tau$, with ρ some function of σ. Consider an ansatz for the azimuthal coordinate

$$\phi = \omega t;$$

this describes a string spinning around the spatial sphere.

(c) Show that with these specifications the Nambu-Goto Lagrangian becomes

$$L = -4\frac{R^2}{2\pi\alpha'}\int_0^{\rho_0} d\rho\sqrt{\cosh^2\rho - (\dot{\phi})^2\sinh^2\rho},$$

where $\coth^2\rho_0 = \omega^2$, and the factor of 4 is because there are four segments of the string stretching from 0 to ρ_0.

Plugging in $t = \tau$, ...

$$\dot{X}^2 = R^2(-\cosh^2\rho + \sinh^2\dot{\phi}^2)$$

$$(X')^2 = R^2(\rho')^2$$

$$\dot{X} \cdot X' = 0.$$

this leaves the action

$$S = -\frac{R^2}{2\pi\alpha'}\int d^2\sigma \sqrt{(\cosh^2\rho - \dot{\phi}^2\sinh^2\rho)(\rho'(\sigma))^2}$$

$$S = -\frac{R^2}{2\pi\alpha'}\int d\tau d\sigma \left|\frac{\partial\rho}{\partial\sigma}\right| \sqrt{\cosh^2\rho - \dot{\phi}^2\sinh^2\rho}.$$

To be closed the string must fold back on itself and traverse $\rho \in [0, \rho_0)$ (ρ_0 is the maximum value of ρ the string achieves) four times (assuming the minimal number of folds). The maximum value of ρ can be determined by demanding that the arg of the sqrt be positive:[3]

$$\coth^2\rho_0 = \omega^2.$$

Restricting the integral to run over four segments where $\rho' > 0$, we can get rid of the absolute value around ρ' use the chain rule, and multiply by 4, and we get the expression given.

(d) Show that the energy and spin of this configuration are

$$E = 4\frac{\sqrt{\lambda}}{2\pi}\int_0^{\rho_0} d\rho \frac{\cosh^2\rho}{\sqrt{\cosh^2\rho - \omega^2\sinh^2\rho}}$$

$$S = 4\frac{\sqrt{\lambda}}{2\pi}\int_0^{\rho_0} d\rho \frac{\omega\sinh^2\rho}{\sqrt{\cosh^2\rho - \omega^2\sinh^2\rho}}$$

[3]Why is this the right thing to do? If we included values of ρ where the sqrt took both signs we would find complex energies which would be hard to interpret. A nice argument that some people like to make is that the 'fold' (*i.e.* the place where ρ' suddenly changes sign) is like a string endpoint (which automatically has Neumann bc's in ρ), and therefore it should move at the speed of light (by the same argument as for Polchinski Problem 1.2 below). This gives the condition that the arg of the sqrt vanishes at the fold. Alternatively again, there are more general solutions where the fold is resolved by allowing the string to move in some other direction at that point, see *e.g.* the nice paper by M. Kruczenski, hep-th/0410226.

where

$$\lambda = \frac{R^4}{\alpha'^2}.$$

Plugging into our expression for the spin, we get

$$S = -\frac{1}{2\pi\alpha'}\int d\sigma \frac{G_{\phi\phi}\dot{\phi}(X')^2}{R^2|\rho'|\sqrt{\cosh^2\rho - \omega^2\sinh^2\rho}} = -\frac{1}{2\pi\alpha'}\int_0^{2\pi} d\sigma \frac{R^4\omega\sinh^2\rho\rho'}{R^2\sqrt{\cosh^2\rho - \omega^2\sinh^2\rho}}$$

$$= -4\frac{R^2}{2\pi\alpha'}\int_0^{\rho_0} d\rho \frac{\omega\sinh^2\rho}{\sqrt{\cosh^2\rho - \omega^2\sinh^2\rho}}.$$

For the energy we get

$$E = -\frac{1}{2\pi\alpha'}\int d\sigma \frac{G_{tt}\dot{t}(X')^2}{R^2|\rho'|\sqrt{\cosh^2\rho - \omega^2\sinh^2\rho}} = \frac{R^2}{2\pi\alpha'}\int_0^{2\pi} d\sigma \frac{\cosh^2\rho\rho'}{\sqrt{\cosh^2\rho - \omega^2\sinh^2\rho}}$$

$$= 4\frac{R^2}{2\pi\alpha'}\int_0^{2\pi} d\sigma \frac{\cosh^2\rho}{\sqrt{\cosh^2\rho - \omega^2\sinh^2\rho}}$$

etc..

Note that I seem to have a sign difference in S relative to GKP. I'm not going to chase it down because it can be undone by reversing the sign of ω.

(e) Next we're going to reproduce this solution from the Polyakov action, in conformal gauge:

$$S_P = \frac{1}{4\pi\alpha'}\int d\tau d\sigma G_{ij}\partial_a X^i \partial^a X^j.$$

In this description, a solution must also satisfy the equations of motion following from varying the worldsheet metric, namely the Virasoro constraints:

$$0 = T_{++} = \partial_+ X^i \partial_+ X^j G_{ij}$$

$$0 = T_{--} = \partial_- X^i \partial_- X^j G_{ij}$$

Show that inserting

$$t = e\tau, \phi = e\omega\tau, \rho = \rho(\sigma)$$

into the Virasoro constraints gives

$$(\rho')^2 = e^2(\cosh^2\rho - \omega^2\sinh^2\rho).$$

(Here e is a bit of slop which can be adjusted to make sure period of σ is 2π.) This leads to an expression for $\frac{d\sigma}{d\rho}$.

We've already seen that

$$X' \cdot \dot{X} = 0$$

automatically in this ansatz. The other Vir constraint is

$$0 = (X')^2 + (\dot{X})^2 = R^2 \left((\rho')^2 + e^2(-\cosh^2\rho + \omega^2 \sinh^2\rho) \right),$$

where $e = \partial_\tau t$ **is a possible mismatch between the two labelling systems. This says**

$$e^2(\cosh^2\rho - \omega^2\sinh^2\rho) = (\rho')^2$$

which means that

$$\frac{d\sigma}{d\rho} = \frac{1}{e\sqrt{\cosh^2\rho - \omega^2\sinh^2\rho}} \qquad (Arthur)$$

Show that the (target-space) energy and spin are

$$E = \frac{R^2}{2\pi\alpha'} e \int_0^{2\pi} d\sigma \cosh^2\rho$$

$$S = \frac{R^2}{2\pi\alpha'} e\omega \int_0^{2\pi} d\sigma \sinh^2\rho.$$

Use your expression for $\frac{d\sigma}{d\rho}$ to show that this reproduces the answers found using the Nambu-Goto action.

From the Noether method,

$$J_a^{spin} = -\frac{1}{2\pi\alpha'} G_{\phi\phi}\partial_a\phi = -\frac{R^2}{2\pi\alpha'}\int_0^{2\pi} d\sigma \;\, e\omega\sinh^2\rho$$

$$J_a^{energy} = -\frac{1}{2\pi\alpha'} G_{tt}\partial_a t = -\frac{R^2}{2\pi\alpha'}\int_0^{2\pi} d\sigma \;\, e\cosh^2\rho$$

Using $(Arthur)$ **to change variables reproduces the expressions from the NG action.**

(f) [**super-bonus challenge**] Expand the integrals for E and S at large spin ($\omega = 1 + 2\eta, \eta \ll 1$), and show that in that regime

$$E - S \sim \sqrt{\lambda} \ln S.$$

The fact that this expression is reminiscent of logarithmic scaling violations in field theory is not a coincidence; the coefficient in front of the log is called the 'cusp anomalous dimension.'

This is a fun exercise in Taylor expansion. A useful fact is that in the limit given $\rho_0 \sim \frac{1}{2}\ln\left(\frac{1}{\eta}\right)$. **This leads to**

$$E = \frac{2\sqrt{\lambda}}{\pi}\int_0^{\rho_0} d\rho \frac{\cosh^2\rho}{\sqrt{1-4\eta\sinh^2\rho}} = \frac{2\sqrt{\lambda}}{\pi}\int_0^{\rho_0} d\rho \cosh^2\rho\left(1+2\eta\sinh^2\rho+\mathcal{O}(\eta^2)\right)$$

etc.. Eventually we get

$$E = \frac{\sqrt{\lambda}}{2\pi}\left(\frac{1}{2\eta}+\ln\left(\frac{1}{\eta}\right)+\mathcal{O}(\eta^0)\right)$$

$$|S| = \frac{\sqrt{\lambda}}{2\pi}\left(\frac{1}{2\eta}-\ln\left(\frac{1}{\eta}\right)+\mathcal{O}(\eta^0)\right)$$

Taking the difference gives

$$E-|S| = \frac{\sqrt{\lambda}}{\pi}\ln\left(\frac{S}{\sqrt{\lambda}}\right)+...$$

Some other problems you might consider doing are

1. **Polchinski Problem 1.2.** Use the Virasoro constraints to show that the endpoints of an open string (with Neumann boundary conditions) move at the speed of light.

   The Virasoro constraints are

   $$X'\cdot\dot{X}=0, \quad (X')^2+\dot{X}^2=0,$$

   which (classically) should hold at every point of the worldsheet. Neumann boundary conditions means that at the spatial boundaries of the worldsheet, the derivative in the normal direction to the boundary vanishes:

   $$\partial_\sigma X^\mu|_{bdy}=0.$$

   Evaluating the second Vir constraint on the boundary, using $(X')^2 = 0$,

   $$0=\dot{X}^2|_{bdy}$$

   which says that $\dot{X}^\mu|_{bdy}$, **the velocity of the endpoint, is a null vector,** *i.e.* **the endpoint moves at the speed of light.**

2. **Polchinski Problem 1.1.** (a) Pick static gauge $\tau = X^0$. Non-relativistic means $\dot{X}^i \equiv v^i \ll 1$. The point-particle action is

$$S_{pp} = -m \int d\tau \sqrt{-\dot{X}^\mu \dot{X}_\mu} = -m \int dt \sqrt{1 - v^2} = \int dt \left(-m + \frac{1}{2} m v^2 + \mathcal{O}(v^4) \right)$$

which clearly displays potential and kinetic terms.

(b) Again we use static gauge, and assume small velocity $\dot{X}^i \equiv v^i \ll 1$. It's convenient to call $w^i \equiv \partial_\sigma X^i$ ('w' is for 'winding'). Then the induced metric is

$$\Gamma_{ab} = \begin{pmatrix} -1 + v^2 & w \cdot v \\ w \cdot v & w^2 \end{pmatrix}.$$

The NG Lagrangian (not density) is then

$$\begin{aligned} L_{NG} = -\frac{1}{2\pi\alpha'} \int d\sigma \sqrt{-\det \Gamma} &= -\frac{1}{2\pi\alpha'} \int d\sigma \sqrt{w^2(1 - v^2) + (v \cdot w)^2} \\ &= -\frac{1}{2\pi\alpha'} \int d\sigma |w| \left(1 - \frac{1}{2} v^2 + \frac{1}{2w^2} (v \cdot w)^2 + \mathcal{O}(v^4) \right) \\ &= -TL + \frac{1}{2} T \int d\sigma |w| (v^2 - \frac{(v\dot{w})^2}{w^2}) + ... \end{aligned}$$

where $L \equiv \int d\sigma |w|$ is the length of the string. The first term here is obviously just the energy from the tension. The second term looks a little funny until we realize that the motion of the string along its extent is a gauge degree of freedom, and introduce the transverse velocity

$$v_T = v - \frac{w \cdot v}{w^2} w.$$

Then we have

$$L_{NG} = -TL + \frac{1}{2} T \int d\sigma |w| v_T^2 + ...$$

and the second term is clearly the kinetic energy of an object with mass TL.