

**Problem Set 8**  
*Worldsheet perspective on CY compactification*

**Due:** Monday, December 18, 2007 at 11:00 AM in the box or in my office.

1. **Orbifolds.**

Suppose you have in your possession a 2d CFT with the correct central charge to provide (part of) a critical string background. Suppose further that this CFT has a symmetry group  $\Gamma$  (for simplicity let's assume it's discrete, finite and abelian). For example, say the CFT is a sigma model, and the target space has a discrete isometry, which could be a subgroup of a continuous isometry.

Let's try to make a new CFT by gauging this symmetry; in particular, this will mean projecting onto states invariant under  $\Gamma$ .

(a) By considering the torus amplitude of this theory, show that a modular transformation ( $\tau \rightarrow -1/\tau$ ) relates the projection onto  $\Gamma$ -invariant states to a sum over new sectors of states called *twisted sectors*. These are sectors of strings which are only closed modulo the action of  $\Gamma$ :

$$x(\sigma + 2\pi) = \gamma x(\sigma)$$

for some  $\gamma \in \Gamma$ .

The resulting procedure of projecting onto invariant states and summing over twisted sectors in a modular-invariant way is called the *orbifold* construction.

(b) Consider the simple example where our initial CFT is four free bosons of infinite radius; group them into complex pairs  $z^{i=1,2}$ , and consider the group  $\Gamma = \mathbb{Z}_N$  generated by

$$\gamma : \begin{pmatrix} z^1 \\ z^2 \end{pmatrix} \rightarrow \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix} \begin{pmatrix} z^1 \\ z^2 \end{pmatrix}, \quad \omega^N = 1.$$

Show that putting a type II string on this CFT preserves half the supersymmetry. (This is a model of part of a K3 surface.)

For the case of  $N = 2$ , compute the massless spectrum. Note that  $\gamma$  does not act freely; the geometric space you get by identifying points under gamma is therefore scary to mathematicians; it is called an  $A_{N-1}$  singularity. Strings propagation on the other hand is completely fine.<sup>1</sup>

(c) [BONUS] Note that the orbifolded theory has a new global discrete symmetry, called the *quantum symmetry* (not my fault), which acts with a phase  $\alpha^k$  on the sector twisted by  $\omega^k$ . This symmetry constrains the interactions of the orbifold theory. What can you say about the theory you get by orbifolding by the quantum symmetry?

## 2. GLSM (Gauged Linear Sigma Model).

A powerful technique for studying nonlinear sigma models on CY manifolds can be found by studying massive (meaning not scale-invariant) 2d QFTs which RG flow to them in the IR. In particular, the parameter space of the massive theories will give some coordinates on the moduli space of the CFT in the IR.<sup>2</sup>

Sigma models with Kähler target have 2d  $(2, 2)$  supersymmetry. So to find good candidate models, let's look at the multiplets of  $(2, 2)$ . The most accessible are are

(i) the chiral multiplet,  $\bar{D}_\alpha \Phi = 0$ , which can be parametrized as

$$\Phi(y, \theta) = \phi(y) + \theta^\alpha \psi_\alpha + \theta^\alpha \theta_\alpha F$$

where  $\phi$  is complex,  $\alpha = \pm$ , and  $F$  is an auxiliary field.

(ii) the vector multiplet,  $V = \bar{V}$ , which contains a vector, gaugino and auxiliary field  $D$ . It can be described in terms of a *twisted* chiral field  $\Sigma \equiv \bar{D}_+ D_- V$ , satisfying  $\bar{D}_+ \Sigma = 0 = D_- \Sigma$ . Its expansion is

$$\Sigma = \sigma + \theta^+ \bar{\lambda}_+ + \bar{\theta}^- \lambda_- + \theta^+ \bar{\theta}^- (D + i v_{01}) + \dots$$

where  $v_{01}$  is the field strength of the  $U(1)$  gauge field.

In two dimensions, even abelian gauge theories confine – the maxwell term is by dimensional analysis a relevant operator. Consider a  $U(1)$  gauge symmetry,

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<sup>1</sup>To be precise, it can be shown that the orbifold CFT has a certain nonzero NSNS  $B_{mn}$  in the background which resolves the singularity. By deforming this to zero it is actually possible to make the perturbative string theory singular. In that limit some wrapped D2-branes become light and can enhance the gauge symmetry (the abelian gauge symmetry coming from the RR vectors) to  $SU(N)$ .

<sup>2</sup>The reference for this discussion is E. Witten, “Phases of  $\mathcal{N} = 2$  theories in two dimensions.” hep-th/9301042.

with  $N + 1$  chiral multiplets of charge 1. An important term in the lagrangian will be the Fayet-Ilioupoulos (FI) term; this is a term

$$= \int d\bar{\theta}_+ d\theta_- \Sigma t = -rD + \theta v_{01}$$

with  $t = r + i\theta$

The whole bosonic potential so far is

$$L_D = \frac{D^2}{2e^2} + D \sum |\phi_i|^2 - Dr.$$

(a) Integrate out the auxiliary field  $D$  to find the bosonic potential. Show that when  $r$  is large and positive (we will denote this regime by the awkward symbol ' $r \gg 0$ '), the  $\phi$ s can't simultaneously vanish and keep zero energy. Convince yourself that the vacuum manifold is therefore  $\mathbb{C}\mathbb{P}^N$ . It is then reasonable to conjecture that this gauge theory flows to a nonlinear sigma model on  $\mathbb{C}\mathbb{P}^N$  with a size related to the FI parameter.

(b) Show that the beta function for the FI parameter is proportional to the sum of the charges of the chiral fields. Relate this to your expectations about the RG flow of the  $\mathbb{C}\mathbb{P}^N$  model.

(c) To get a CY threefold, take  $N = 5$  above, and add one more field  $P$  of charge  $-5$ . Note that the FI parameter no longer runs. Now we can add a gauge-invariant superpotential

$$W = PG_5(\Phi).$$

Find the bosonic potential. In the case where  $G$  is transverse, show that the vacuum manifold is  $\{P = 0\}$  and

$$\{G_5(\phi)\} \in \mathbb{C}\mathbb{P}^4.$$

The fluctuations perpendicular to this locus are all massive, and the IR limit seems to give a sigma model on the quintic. Relate the moduli of the quintic to the marginal parameters of the gauge theory.

(d) Consider varying the complex structure moduli (via the coefficients of  $G_5$ ) so that  $G_5$  is not transverse, *i.e.* where  $G = dG = 0$  *does* have a solution for  $\phi$  not all zero. What happens to the vacuum manifold in this case?

(e) There's another semiclassical limit where  $|r|$  is large, namely  $r \ll 0$ . Convince yourself that the  $D$  term condition implies that  $P$  must have an expectation value in this limit. Show that the resulting effective theory is a *Landau-Ginzburg orbifold*, a model of massless fields with a superpotential

$$W \propto G_5(\phi)$$

orbifolded by a  $\mathbb{Z}_5$  symmetry. This is a completely non-geometric CFT, connected by variation of Kähler moduli to the quintic NLSM.

(f) Remember that the vectormultiplet contains a complex scalar  $\sigma$ . Under the relation to 4d  $\mathcal{N} = 1$ , these modes related to the components of the 4d gauge field along the two dimensions along which we reduced. This tells us that there must be potential terms of the form

$$V \supset |\sigma|^2 \left( \sum_i |\phi_i|^2 + 25|p|^2 \right).$$

When  $r \gg 0$ ,  $\sum |\phi_i|^2 \sim r$  means that  $\sigma$  gets a large mass; similarly when  $r \ll 0$ ,  $|p|^2 \sim r$  gives  $\sigma$  a large mass. What happens in between? Naively at  $r = 0$ , we can set  $\phi_i = p = 0$  consistent with  $D = 0, dW = 0$ , so it would seem that  $\sigma$  becomes a flat direction there! Such an extra branch of the target space of the sigma model at a point in its parameter space would lead the correlators of the CFT to diverge there.<sup>3</sup>

(g) We saw above in section (d) how to make a GLSM for an orbifold. Consider the GLSM with one  $U(1)$  and three chiral fields  $(Z^1, Z^2, P)$  of charges  $(1, N - 1, -N)$ . Show that the FI parameter is marginal. Show that when  $r \ll 0$ , this flows to the orbifold of  $\mathbb{C}^2$  studied in problem 1. Interpret the variation of the FI parameter as condensing a mode of the twisted sector (this is called *blowing up the singularity*). In the case  $N = 2$ , show that in the space at  $r \gg 0$ , the orbifold point has been replaced by a  $\mathbb{P}^1$ . It is this 2-sphere that shrinks in making the  $A_1$  singularity, and this is why a D2-brane can become massless there.

(i) [BONUS] Compute the Witten index of the gauge theory with  $N + 1$  chiral fields based on our answer for the vacuum manifold at large radius. This means that there are at least  $N + 1$  supersymmetric vacua. What happens to them in the IR when the  $\mathbb{C}\mathbb{P}^N$  shrinks to nothing?

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<sup>3</sup>Actually the correlators only diverge if the theta angle is also zero, because otherwise there is a constant electric field in the vacuum.