University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215A QFT Fall 2023
Assignment 10 ("Final Exam")

Due 11:59pm Wednesday, December 13, 2023

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1. **Brain-warmers.**

   (a) Check that
   $$(p \cdot \sigma)\,(p \cdot \bar{\sigma}) = p^2.$$

   (b) Use the previous part to show that if
   $$u_r(\vec{p}) = \begin{pmatrix} \sqrt{p \cdot \sigma}\,\xi_r \\ \sqrt{p \cdot \bar{\sigma}}\,\xi_r \end{pmatrix} \quad \text{and} \quad v_r(\vec{p}) = \begin{pmatrix} \sqrt{p \cdot \sigma}\,\eta_r \\ -\sqrt{p \cdot \bar{\sigma}}\,\eta_r \end{pmatrix}$$
   with $p^2 = m^2$ (solutions of the Dirac equation with mass m), then
   $$\bar{u}_r(\vec{p})u_s(\vec{p}) = 2m\xi_r^\dagger \xi_s \quad \text{and} \quad \bar{v}_r(\vec{p})v_s(\vec{p}) = -2m\eta_r^\dagger \eta_s$$
   (where $\bar{u} \equiv u^\dagger \gamma^0$ as usual).

   (c) Show that $\bar{u}_r(\vec{p})v_s(\vec{p}) = 0$ and $u_r(\vec{p})^\dagger v_s(-\vec{p}) = 0$ but $u_r(\vec{p})^\dagger v_s(\vec{p}) \neq 0$.

2. **Other bases for gamma matrices.** [Bonus problem]

   Many different bases of gamma matrices are frequently used by humans. You may read on the internet someone telling you that the gamma matrices are
   $$\tilde{\gamma}^0 = \begin{pmatrix} \mathbb{1}_{2\times 2} & 0 \\ 0 & -\mathbb{1}_{2\times 2} \end{pmatrix}, \quad \tilde{\gamma}^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$
   and think that I have lied to you. This basis is useful for studying the non-relativistic limit. The Weyl basis which we introduced in lecture instead makes manifest the reducibility of the Dirac spinor into L plus R Weyl spinors. Find the unitary matrix U which relates them $\tilde{\gamma}^\mu = U\gamma^\mu U^\dagger$.

3. **Symmetries of the Dirac lagrangian.**

   (a) Find the Noether currents j^μ and j_5^μ associated with the transformations $\Psi(x) \to e^{-\mathbf{i}\alpha}\Psi(x)$ and $\Psi(x) \to e^{-\mathbf{i}\alpha\gamma^5}\Psi(x)$ of a free Dirac field. Show by explicit calculation that the former is conserved and the latter is conserved (at least classically) if $m = 0$.

(b) Find the conserved currents associated with the Lorentz symmetry $\Psi(x) \mapsto \Lambda_{\frac{1}{2}}(\theta, \beta)\Psi(\Lambda^{-1}x)$ of the Dirac Lagrangian. Show that the conserved charge takes the form

$$J^{\mu\nu} = \int_{\text{space}} \left(\mathcal{J}^{\mu\nu}_{\text{orbital}} + \Psi^\dagger J^{\mu\nu}_{\text{Dirac}} \Psi \right)$$

where $\mathcal{J}^{\mu\nu}_{\text{orbital}}$ has the form it would have for a scalar field, and $J^{\mu\nu}_{\text{Dirac}} \equiv \frac{\mathbf{i}}{4}[\gamma^\mu, \gamma^\nu]$ are the matrices satisfying the Lorentz algebra.

Convince yourself that the latter matrix specifies how the current acts in the one-particle sector.

4. **Meson scattering.**

Consider the Yukawa theory with fermions, with $\mathcal{L}_{\text{int}} = -g\bar{\Psi}\Psi\phi$, where Ψ is a Dirac fermion field and ϕ is a real scalar field.

(a) Draw a Feynman diagram that gives a leading contribution to the scattering amplitude for the process $\phi\phi \to \phi\phi$.

(b) [Bonus problem] Derive the correct sign of the amplitude by considering the relevant matrix elements of powers of the interaction hamiltonian.

(c) Evaluate the diagram in terms of a spinor trace and a momentum integral. Do not do the momentum integral. Suppose that the integral is cut off at large k by some cutoff Λ. Estimate the dependence on Λ.

5. **The magnetic moment of a Dirac fermion.**

In this problem we consider the hamiltonian density

$$\mathfrak{h}_I = q\bar{\Psi}\gamma^\mu\Psi A_\mu \ .$$

This describes a local, Lorentz invariant, and gauge invariant interaction between a Dirac fermion field Ψ and a vector potential A_μ. In this problem, we will treat the vector potential, representing the electromagnetic field, as a fixed, classical background field.

Define single-particle states of the Dirac field by $\langle 0|\,\Psi(x)\,|\vec{p}, s\rangle = e^{-\mathbf{i}px}u^s(p)$. We wish to show that these particles have a magnetic dipole moment, in the sense that in their rest frame, their (single-particle) hamiltonian has a term $h_{NR} \ni \mu_B \vec{S} \cdot \vec{B}$ where $\vec{S} = \frac{1}{2}\vec{\sigma}$ is the particle's spin operator.

(a) q is a real number. What is required of A_μ for $H_I = \int d^3x \mathfrak{h}_I$ to be hermitian?

(b) [Bonus problem] How must A_μ transform under parity P and charge conjugation C in order for H_I to be invariant? (To answer this, you'll have to find out how the spinor bilinear transforms, *e.g.* from Peskin.) How do the electric and magnetic fields transform? Show that this allows for a magnetic dipole moment but not an electric dipole moment.

(c) Show that in the non-relativistic limit

$$\bar{u}(p')\gamma^{\mu\nu}u'(p)F_{\mu\nu} = a\xi^\dagger\sigma \cdot \vec{B}\xi'$$

for some constant a (find a). Recall that $\gamma^{\mu\nu} \equiv \frac{1}{2}[\gamma^\mu, \gamma^\nu]$. Here u, u' are positive-energy solutions of the Dirac equation with mass m and

$$u \overset{NR}{\to} \sqrt{m}(\xi, \xi), u' \overset{NR}{\to} \sqrt{m}(\xi', \xi')$$

in the non-relativistic limit.

(d) Suppose that A_μ describes a magnetic field $\vec{B}$ that is uniform in space and time.

Show that in the non-relativistic limit

$$\langle \vec{p}', s'| H_I |\vec{p}, s\rangle = \not{\delta}^3 (\vec{p} - \vec{p}')\, h(\xi, \xi', \vec{B}) + ...$$

where ... is terms independent of the spin. Find the function $h(\xi, \xi', \vec{B})$. You may wish to use the Gordon identity. Rewrite the result in terms of single-particle states with non-relativistic normalization (*i.e.* $\langle \vec{p}|\vec{p}'\rangle_{NR} = \not{\delta}^3(p - p')$). Interpret h as a non-relativistic hamiltonian term saying that the gyromagnetic ratio of the electron is $-g\frac{|q|}{2m}$ with $g = 2$.

(e) [optional] How does the result change if we add the term

$$\Delta H = \frac{c}{M}\bar{\Psi}F_{\mu\nu}[\gamma^\mu, \gamma^\nu]\Psi \quad ?$$

6. **Non-relativistic interactions from QFT.**

(a) **Coulomb potential.**

Derive from QED that the force between non-relativistic electrons is a repulsive $1/r^2$ force law!

(b) **Pseudoscalar Yukawa theory.**

Consider the theory of a massive Dirac fermion Ψ and a massive pseudoscalar φ interacting via the term

$$V_5 \equiv g_5\bar{\Psi}\gamma^5\Psi\varphi.$$

Convince yourself that this theory is parity invariant (for some assignment of the action of parity on the fields).

List the Feynman rules.

Draw and evaluate the diagrams contributing to $\Psi\Psi \to \Psi\Psi$ scattering at leading order in g_5.

Consider the non-relativistic limit, $m \gg |\vec{p}|$ and find the effective interaction hamiltonian. If you happen to find zero for the leading term, then it's not the leading term.

7. **Equivalent photon approximation.** [Bonus problem] Consider a process in which very-high-energy electrons scatter off a target. At leading order in α, the electron line is connected to the rest of the diagram by a single photon propagator. If the initial and final energies of the electron are E and E', the photon will carry momentum q with $q^2 = -2EE'(1-\cos\theta)$ (ignoring the electron mass $m \ll E$). In the limit of forward scattering ($\theta \to 0$), we have $q^2 \to 0$, so the photon approaches its mass shell. In this problem, we ask: To what extent can we treat it as a real photon?

   (a) The matrix element for the scattering process can be written as

   $$\mathcal{M} = -\mathbf{i}e\bar{u}(p')\gamma^\mu u(p)\frac{-\mathbf{i}\eta_{\mu\nu}}{q^2}\hat{\mathcal{M}}^\nu(q)$$

   where $\hat{\mathcal{M}}^\nu$ represents the coupling of the virtual photon to the target. Let $q = (q^0, \vec{q})$ and define $\tilde{q} = (q^0, -\vec{q})$. The contribution to the amplitude from the electron line can be parametrized as

   $$\bar{u}(p')\gamma^\mu u(p) = Aq^\mu + B\tilde{q}^\mu + C\epsilon_1^\mu + D\epsilon_2^\mu$$

   where ϵ_α are unit vectors transverse to $\vec{q}$. Show that B is at most of order θ^2 (dot it with q), so we can ignore it at leading order in an expansion about forward scattering. Why do we not care about the coefficient A?

   (b) Working in the frame with $p = (E, 0, 0, E)$, compute

   $$\bar{u}(p')\gamma \cdot \epsilon_\alpha u(p)$$

   explicitly using massless electrons, where $\bar{u}$ and u are spinors of definite helicity, and $\epsilon_{\alpha=\parallel,\perp}$ are unit vectors parallel and perpendicular to the plane of scattering. Keep only terms through order θ. Note that for $\epsilon_\parallel$, the (small) $\hat{3}$ component matters.

   (c) Now write the expression for the electron scattering cross section, in terms of $|\hat{\mathcal{M}}^\mu|^2$ and the integral over phase space of the target. This expression must be integrated over the final electron momentum $\vec{p}'$. The integral over $p^{3'}$ is an integral over the energy loss of the electron. Show that the integral over $p'_\perp$ diverges logarithmically as $p'_\perp$ or $\theta \to 0$.

(d) The divergence as $\theta \to 0$ is regulated by the electron mass (which we've ignored above). Show that reintroducing the electron mass in the expression

$$q^2 = -2(EE' - pp' \cos\theta) + 2m^2$$

cuts off the divergence and gives a factor of $\log(s/m^2)$ in its place.

(e) Assembling all the factors, and assuming that the target cross sections are independent of photon polarization, show that the largest part of the electron-target cross section is given by considering the electron to be the source of a beam of real photons with energy distribution given by

$$N_\gamma(x)dx = \frac{dx}{x}\frac{\alpha}{2\pi}(1 + (1-x)^2)\log\frac{s}{m^2}$$

where $x \equiv E_\gamma/E$. This is the Weiszäcker-Williams equivalent photon approximation. It is a precursor to the theory of jets and partons in QCD.

8. **Electron-positron scattering.** [Bonus problem]

Draw and evaluate the two diagrams that contribute to $e^+e^- \to e^+e^-$ (Bhabha) scattering at tree level in QED. Be careful about the relative sign of their contributions.

Compare with the case of $e^+e^- \to \mu^+\mu^-$ and with $e^-e^- \to e^-e^-$.

9. **Supersymmetry.** [Bonus problem] A continuous symmetry that mixes bosons and fermions is called *supersymmetry*.

(a) The simplest example of a supersymmetric field theory is the theory of a free complex boson and a free Weyl fermion, with Lagrangian is

$$\mathcal{L} = \partial_\mu\phi^\star\partial^\mu\phi + \chi^\dagger \mathbf{i}\bar{\sigma}^\mu\partial_\mu\chi + F^\star F.$$

Here F is an auxiliary field whose purpose is to make the supersymmetry transformations look nice. Show that the action is invariant under

$$\delta\phi = -\mathbf{i}\epsilon^T\sigma^2\chi, \delta\chi = \epsilon F + \sigma\cdot\partial\phi\sigma^2\epsilon^\star, \delta F = -\mathbf{i}\epsilon^\dagger\bar{\sigma}\cdot\partial\chi \tag{1}$$

where the symmetry parameter ϵ is a 2-component spinor of Grassmann numbers.

(b) Show that the term

$$\Delta\mathcal{L} = \left(m\phi F + \frac{1}{2}\mathbf{i}m\chi^T\sigma^2\chi\right) + h.c.$$

is also invariant under the transformation (1). Eliminate F from the full Lagrangian $\mathcal{L} + \Delta\mathcal{L}$ by solving its equations of motion, and show that the fermion and boson fields are given the same mass.

(c) We can include supersymmetric interactions as well. Show that the following field theory is supersymmetric:

$$\mathcal{L} = \partial_\mu \phi_i^\star \partial^\mu \phi^i + \chi_i^\dagger \mathbf{i} \bar{\sigma} \cdot \partial \chi_i + F_i^\star F_i + \left(F_i \partial_{\phi_i} W + \frac{\mathbf{i}}{2} \partial_{\phi_i} \partial_{\phi_j} W \chi_i^T \sigma^2 \chi_j + h.c. \right)$$

where $i = 1..n$ and $W = W(\phi)$ is an arbitrary function of the ϕ_i, called the *superpotential*. For the simple case $n = 1$ and $W = g\phi^3/3$ write out the field equations for ϕ and χ after eliminating F.