University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215A QFT Fall 2023
Assignment 5

Due 11:00am Monday, November 6, 2023

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1. **The iϵ prescription produces time-ordered correlators.** [Bonus problem]

   Check that the four-point function of a free scalar field of mass m

   $$Z^{-1} \int [D\phi] e^{\mathbf{i}S[\phi]} \prod_{i=1}^{4} \phi(x_i)|_{m^2 \to m^2 - \mathbf{i}\epsilon},$$

   defined by the $\mathbf{i}\epsilon$ prescription, is the *time-ordered* vacuum expectation value

   $$= \langle 0| \mathcal{T}\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) |0\rangle .$$

   One way to do this is to show that they are both equal to

   $$D_T(12)D_T(34) + D_T(13)D_T(24) + D_T(14)D_T(23), \tag{1}$$

   where $D_T(ij) \equiv \langle 0| \mathcal{T}\phi(x_i)\phi(x_j) |0\rangle$.

2. **The propagator is a Green's function.**

   (a) Consider the *retarded* propagator for a real, free, massive scalar field:

   $$D_R(x-y) \equiv \theta(x^0 - y^0) \langle 0| [\phi(x), \phi(y)] |0\rangle .$$

   Show that it is a Green's function for the Klein-Gordon operator, in the sense that

   $$\left(\Box_x + m^2\right) D_R(x-y) = a\delta^{d+1}(x-y), \qquad \Box_x \equiv \frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\mu}.$$

   Find a.

   (b) Use the previous result to generalize the mode expansion of the scalar field to the situation with an external *source*, *i.e.* when we add to the lagrangian density $\phi(x)j(x)$ for some fixed c-number function $j(x)$.

   (c) Show that the *time-ordered* propagator for a real, free, massive scalar field

   $$D(x-y) \equiv \langle 0| \mathcal{T}\phi(x)\phi(y) |0\rangle \equiv \theta(x^0-y^0) \langle 0| \phi(x)\phi(y) |0\rangle + \theta(y^0-x^0) \langle 0| \phi(y)\phi(x) |0\rangle$$

is *also* a Green's function for the Klein-Gordon operator. That is, consider what happens when you act with the wave operator $+\Box_x + m^2$ on the time-ordered two-point function.

[Hints: Use the canonical equal-time commutation relations:

$$[\phi(\vec{x}), \phi(\vec{y})] = 0, \quad [\partial_{x^0}\phi(\vec{x}), \phi(\vec{y})] = -\mathbf{i}\delta^{D-1}(\vec{x}-\vec{y}).$$

Do not neglect the fact that $\partial_t \theta(t) = \delta(t)$: the time derivatives act on the time-ordering symbol!]

(d) This is an example of something that is easier to understand from the path integral. In the next problem, we'll understand why the correlation functions of ϕ should solve the equations of motion, up to 'contact terms'.

3. **Schwinger-Dyson equations.**

Consider the path integral

$$\int [D\phi] e^{\mathbf{i}S[\phi]}.$$

Using the fact that the integration measure is independent of the choice of field variable, we have

$$0 = \int [D\phi] \frac{\delta}{\delta\phi(x)} \text{(anything)}$$

(as long as 'anything' doesn't grow too fast at large ϕ). So this equation says that we can integrate by parts in the functional integral.

(Why is this true? As always when questions about functional calculus arise, you should think of spacetime as discrete and hence the path integral measure as simply the product of integrals of the field value at each spacetime point, $\int [D\phi] \equiv \int \prod_x d\phi(x)$, this is just the statement that

$$0 = \int d\phi_x \frac{\partial}{\partial \phi_x} \text{(anything)}$$

with $\phi_x \equiv \phi(x)$, *i.e.* that we can integrate by parts in an ordinary integral if there is no boundary of the integration region.)

This trivial-seeming set of equations (we get to pick the 'anything') can be quite useful and they are called Schwinger-Dyson equations (or sometimes Ward identities). Unlike many of the other things we'll discuss, they are true non-perturbatively, *i.e.* are really true, even at finite coupling. They provide a quantum implementation of the equations of motion.

(a) Evaluate the RHS of

$$0 = \int [D\phi] \frac{\delta}{\delta\phi(x)} \left(\phi(y) e^{\mathbf{i}S[\phi]}\right)$$

(defined by the $\mathbf{i}\epsilon$ prescription) to conclude that

$$\frac{1}{Z} \int [D\phi] e^{\mathbf{i}S} \frac{\delta S}{\delta\phi(x)} \phi(y) = +\mathbf{i}\delta(x-y). \tag{2}$$

(b) These Schwinger-Dyson equations are true in interacting field theories; to get some practice with them we consider here a free theory. Evaluate (2) for the case of a free massive real scalar field to show that the (two-point) time-ordered correlation functions of ϕ satisfy the equations of motion, most of the time. That is: the equations of motion are satisfied away from other operator insertions:

$$\left(+\Box_x + m^2\right) \langle \mathcal{T} \phi(x)\phi(y) \rangle = -\mathbf{i}\delta(x-y), \tag{3}$$

with $\Box_x \equiv \partial_{x^\mu}\partial^{x^\mu}$.

(c) Find the generalization of (3) satisfied by (time-ordered) three-point functions of the free field ϕ.

4. **QFT in $0+0$ dimensions.**

In this problem we return to the simplest scalar QFT, namely a single integral with quartic action.

(a) By a change of integration variable show that

$$Z = \int_{-\infty}^{\infty} dq \; e^{-S(q)}$$

with $S(q) = \frac{1}{2}m^2q^2 + gq^4$ is of the form

$$Z = \frac{1}{g^{1/4}} \mathcal{Z}\left(m^2/\sqrt{g}\right).$$

This means you can make your life easier by setting $g = 1$, without loss of generality.

(b) Convince yourself (*e.g.* with Mathematica) that the integral really is expressible as a Bessel function.

(c) It would be nice to find a better understanding for why the partition function of (0+0)-dimensional ϕ^4 theory is a Bessel function. Find a Schwinger-Dyson equation for this system which has the form of Bessel's equation for

$$K(x^2/a) \equiv e^{-x^2/a}(x^2)^{-1/4}\mathcal{Z}(x)$$

for some constant a. (If you get stuck I can tell you what function to choose for the 'anything' in the S-D equation.)

(d) Make a plot of the perturbative approximations to the 'Green function' $G \equiv \langle q^2 \rangle$ as a function of g, truncated at orders 1 through 6 or so. Plot them against the exact answer.

(e) (Bonus problem) Show that $c_{n+1} \sim -\frac{2}{3}nc_n$ at large n, where c_n is the coefficient of g^n in either G or Z (by brute force or by cleverness).

5. **Diagrams in 0+0d field theory.** [Bonus problem] Here we will study a bit more some field theories with no dimensions at all, that is, integrals.

Consider the case where we put a label on the 'field': $q \to q_a, a = 1..N$. So we are studying

$$Z = \int \int_{-\infty}^{\infty} \prod_a dq_a \; e^{-S(q)}.$$

Let

$$S(q) = \frac{1}{2} q_a K_{ab} q_b + T_{abcd} q_a q_b q_c q_d$$

where T_{abcd} is a collection of couplings. Assume K_{ab} is a real symmetric matrix.

(a) Show that the propagator has the form:

$$a - - - - - - b \equiv \langle q_a q_b \rangle_{T=0} = \left(K^{-1}\right)_{ab} = \sum_k \phi_a(k)^\star \frac{1}{k} \phi_b(k)$$

where $\{k\}$ are the eigenvalues of the matrix K and $\phi_a(k)$ are the eigenvectors in the a-basis.

(b) Develop a diagrammatic expansion for the propagator $\langle q_a q_b \rangle$. Show that in a diagram with a loop, we must sum over the eigenvalue label k. (For definiteness, consider the order-T correction to the propagator $\langle q_k q_{k'} \rangle$, where k, k' label eigenvectors of K, and $q_k \equiv \sum_a \phi_a(k)^\star q_a$).

(c) Consider the case where $K_{ab} = t\left(\delta_{a,b+1} + \delta_{a+1,b}\right)$, with periodic boundary conditions: $a + N \equiv a$. Find the eigenvalues. Show that in this case if

$$T_{abcd} q_a q_b q_c q_d = \sum_a g q_a^4$$

the k-label is conserved at vertices, *i.e.* the vertex is accompanied by a delta function on the sum of the incoming eigenvalues.

(d) (Bonus question) What is the more general condition on T_{abcd} in order that the k-label is conserved at vertices?

(e) (Bonus question) Study the physics of the model described in 5c.

6. **The vacuum is a fluid with $p = -\rho$.** [Bonus problem]

We said in lecture that the vacuum energy density ρ gravitates and that, when positive, its effect is to cause space to inflate – to expand exponentially in time. An important aspect of this phenomenon is that the vacuum fluctuations produce not only an energy density, but a *pressure*, $p = T^i_i$ (no sum on i), of the form $p = -\rho$, which is negative for $\rho > 0$. The vacuum therefore acts as a perfect fluid with $P = -\rho$. (The stress tensor for a perfect fluid in terms of its velocity field u^μ takes the form $T^{\mu\nu} = (p+\rho)u^\mu u^\nu + pg^{\mu\nu}$, so in a frame with $u^\mu = (1, \vec{0}^\mu)$, $T^0_0 = \rho, T^i_i = P$.) Solving Einstein's equations with such a source produces an inflating universe. In this problem we show that this is is the case.

(a) Show that the energy-momentum tensor for a free relativistic scalar field ($S[\phi] = \int d^D x \sqrt{g}\mathcal{L}, \mathcal{L} = \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{m^2}{2}\phi^2$) takes the form

$$T_{\mu\nu} = a\partial_\mu\phi\partial_\nu\phi - bg_{\mu\nu}\mathcal{L}$$

with some constants a, b.

(b) Reproduce the formal expression for the vacuum energy

$$\langle 0|\mathbf{H}|0\rangle = V\int đ^d k \frac{1}{2}\hbar\omega_{\vec{k}}$$

using the two point function

$$\langle 0|\,\phi(x)^2\,|0\rangle = \langle 0|\phi(0)\phi(0)|0\rangle = \lim_{\vec{x},t\to 0}\langle 0|\,\phi(x)\phi(0)\,|0\rangle$$

and its derivatives. (V is the volume of space.)

(We will learn to draw this amplitude as a Feynman diagram which is a circle (a line connecting a point to itself).)

(c) Show that the vacuum expectation value of the pressure

$$\langle 0|T_{ii}|0\rangle$$

(no sum on i) gives the same answer, up to a sign.

[Hints: You'll find a quite different looking integral from the vacuum energy. Use rotation invariance of the vacuum to simplify the answer. The claim is that however you regulate the integral for the vacuum pressure and $\frac{1}{2}\int đ^d k\omega_k$, you'll get the same answer. A convenient regulator is *dimensional regularization*: treat the dimension d as an arbitrary complex number.]

(d) Show that the resulting vacuum energy momentum tensor ($T_{00} = \rho, T_{ii} = -\rho$ (no sum on i)) is the same as the contribution to the energy-momentum tensor from an action of the form

$$S_{\mathrm{cc}} = \int d^D x \sqrt{g} \Lambda$$

where Λ is a constant (the cosmological constant).

If you wish, plug in the FRW ansatz for the metric $ds^2 = -dt^2 + a(t)^2 d\vec{x}^2$ and show that Einstein's equations in the presence of a positive cosmological constant

$$\frac{\delta S[g]}{\delta g_{\mu\nu}(x)} = 0, \text{ with } S[g] = \frac{1}{16\pi G_N} \int d^D x \sqrt{g} R + S_{\mathrm{cc}} \tag{4}$$

have the solution $a(t) = e^{Ht}$ for some H determined by Λ and G_N.

(e) Argue that $p = -\rho$ is required in order that the vacuum energy does not specify a preferred rest frame.