University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215A QFT Fall 2022
Assignment 8

Due 11:00am Thursday, November 17, 2022

---

1. **Brain warmer.** Mandelstam variables are Lorentz-invariant combinations of the kinematic variables in $2 \leftarrow 2$ scattering

$$s = (p_1 + p_2)^2, t = (p_1 - p_3)^2, u = (p_1 - p_4)^2$$

where p_i^μ are on-shell four-vectors $p_i^2 = m_i^2$ satisfying overall momentum conservation $p_1 + p_2 = p_3 + p_4$. Show that they are not independent but rather satisfy the relation

$$s + t + u = \sum_i m_i^2$$

where i runs over the four external particles.

2. **Particle creation by an external source, continued.**

Consider again the Hamiltonian

$$H = H_0 + \int d^3x \left(-j(t, \vec{x})\phi(x)\right)$$

where H_0 is the free Klein-Gordon Hamiltonian, ϕ is the Klein-Gordon field, and j is a c-number scalar function.

(a) Compute the probability amplitude for the source to create one particle of momentum k. Perform this computation first to $\mathcal{O}(j)$, and then to all orders, using the trick from HW07 to sum the series.

Compute the analogous amplitude for n particles of definite momenta.

(b) Show that the probability of producing n particles (of any momenta) is given by the Poisson distribution,

$$P(n) = \frac{1}{n!}\lambda^n e^{-\lambda}.$$

[Note that for $n > 1$, this requires the measure for the final state phase space.]

(c) Prove the following facts about the Poisson distribution:

$$\sum_{n=0}^{\infty} P(n) = 1, \quad \langle N \rangle \equiv \sum_{n=0}^{\infty} n P(n) = \lambda,$$

that is, $P(n)$ is a probability distribution, and $\langle N \rangle = \lambda$ as predicted. Compute the fluctuations in the number of particles produced $\langle (N - \langle N \rangle)^2 \rangle$.

3. Can there ever be a resonance in a t-channel diagram? Let me break the question down a bit:

(a) Consider a $2 \leftarrow 2$ scattering process where all the particles have the same mass. Let p_1, p_2 be the momenta of the particles in the initial state. Prove that the Mandelstam variables $t = (p_1 - p_3)^2$ and $u = (p_1 - p_4)^2$ cannot be positive when the particles are on-shell $p_i^2 = m^2$.

(b) Bonus problem: What happens if the particles have different masses? It may be worth distinguishing two cases:
(a) when the collision is *elastic*, so that the particles retain their identity and therefore $m_1 = m_3$ and $m_2 = m_4$.
(b) the fully general case where m_i are all different.