

Physics 215A QFT Fall 2022 Assignment 6

Due 11:00am Thursday, November 3, 2022

1. Recovering non-relativistic quantum mechanics.

Consider a complex scalar field, in the non-relativistic limit,

$$\Phi = \sqrt{2m}e^{-imt}\Psi, \quad |\dot{\Psi}| \ll m\Psi.$$

Recall that in this limit, the antiparticles disappear and the mode expansion is

$$\Psi(x) = \int d^d p e^{+i\vec{p}\cdot\vec{x}} \mathbf{a}_p, \quad \Psi^\dagger(x) = \int d^d p e^{-i\vec{p}\cdot\vec{x}} \mathbf{a}_p^\dagger.$$

(a) Show that

$$\hat{P}_i \equiv \int d^d p p_i \mathbf{a}_p^\dagger \mathbf{a}_p$$

is the generator of translations and commutes with the Hamiltonian.

(b) Let

$$\hat{X}^i \equiv \int d^d x \Psi^\dagger(x) x^i \Psi(x).$$

A state of one particle at location \vec{x} is

$$|x\rangle = \Psi^\dagger(x) |0\rangle.$$

Show that

$$\hat{X}^i |x\rangle = x^i |x\rangle.$$

(c) Consider the general one-particle state

$$|\psi\rangle = \int d^d x \psi(x) \Psi^\dagger(x) |0\rangle = \int d^d x \psi(x) |x\rangle.$$

Show that

$$\hat{X}^i |\psi\rangle = \int d^d x x^i \psi(x) |x\rangle$$

and (a little more involved)

$$\hat{P}^i |\psi\rangle = \int d^d x \left(-i \frac{\partial}{\partial x^i} \psi(x) \right) |x\rangle,$$

which is the usual action of these operators on single-particle wavefunctions $\psi(x)$.

2. Scalar Yukawa amplitudes.

Consider again the scalar Yukawa theory of a complex scalar Φ and a real scalar ϕ . In the following, assume all particles are in momentum eigenstates. Use artisanal methods.

- (a) Compute the amplitude for the annihilation of a Φ particle and a Φ^* particle into a ϕ particle, at leading order in the coupling g .
- (b) Compute the amplitude for $\Phi + \phi \rightarrow \Phi + \phi$ scattering to the leading non-trivial order in the coupling.

3. Wick example.

For a real scalar field, verify by hand Wick's prediction for the difference

$$\mathcal{T}(\phi(x_1)\phi(x_2)\phi(x_3)) - : \phi(x_1)\phi(x_2)\phi(x_3) :$$

4. **Fields and forces.** [from Banks] Consider a real free relativistic scalar field of mass m $S[\phi] = \int d^{d+1}x \frac{1}{2} (\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2)$.

- (a) Calculate the vacuum expectation value

$$\langle 0 | \mathcal{T} \left(e^{i \int d^{d+1}x \phi(x) J(x)} \right) | 0 \rangle \equiv e^{iW[J]}$$

where J is a fixed, external (c-number) source. Use Wick's theorem. Make a series expansion in powers of J and draw some diagrams. To understand the structure of the series, recall the formula on a previous homework for $\langle e^{K \cdot q} \rangle$ in any gaussian theory.

- (b) Now specialize to the case where the source is static and is present for a time $2T$:

$$J(x) = J_{\text{static}} \equiv \theta(T-t)\theta(t+T) (\delta^d(x) - \delta^d(x-R))$$

with $T \gg R \gg 1/m$. You should find an answer of the form

$$W[J_{\text{static}}(x)] = TV(R)$$

where $V(R)$ is the Yukawa potential.

- (c) Chant the following incantation:

Static sources experience a force due to exchange of virtual particles.

Feel happy at having reproduced by canonical methods the answer we found earlier using path integral methods.