University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215A QFT Fall 2022
Assignment 3

Due 11:59pm Thursday, October 13, 2021

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1. **Classical Maxwell theory.** [Peskin problem 2.1, lightly edited] Classical electromagnetism follows from the action

$$S[A] = \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^\mu A_\mu \right), \quad \text{where } F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu .$$

   (a) Derive Maxwell's equations as the Euler-Lagrange equations of this action, treating the components $A_\mu(x)$ as the dynamical variables

$$0 = \frac{\delta S[A]}{\delta A_\mu(x)} .$$

   Write the equations in the standard form by identifying $E^i = -F^{0i}$ and $\epsilon^{ijk} B^k = -F^{ij}$.

   (b) Construct the energy-momentum tensor for this theory, when $j^\mu = 0$. Note that the usual procedure

$$T^\mu_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \mathcal{L} \delta^\mu_\nu$$

   does not result in a symmetric tensor. (It is also not gauge invariant.) To remedy that, we can add to $T^{\mu\nu}$ a term of the form $\partial_\lambda K^{\lambda\mu\nu}$, where $K^{\lambda\mu\nu}$ is antisymmetric in its first two indices. Such an object is automatically divergenceless, so

$$\widehat{T}^{\mu\nu} \equiv T^{\mu\nu} + \partial_\lambda K^{\lambda\mu\nu}$$

   is an equally good energy-momentum tensor with the same globally conserved energy and momentum. Show that this construction, with

$$K^{\lambda\mu\nu} = F^{\mu\lambda} A^\nu ,$$

   leads to an energy-momentum tensor $\widehat{T}$ that is symmetric and yields the standard formulae for the electromagnetic energy and momentum densities:

$$\mathcal{E} = \frac{1}{2}\left(E^2 + B^2\right), \quad \vec{S} = \vec{E} \times \vec{B} .$$

(c) [Bonus problem] A better way to think about the energy-momentum tensor is to regard it as the response to a change in the background metric. (This is why it appears as a source in Einstein's equations.) To couple the Maxwell theory to a general background metric $g_{\mu\nu}$, we replace all the $\eta_{\mu\nu}$s with $g_{\mu\nu}$s:

$$S[A,g] = \int d^4x \sqrt{g} \left(-\frac{1}{4} F_{\mu\nu} F_{\rho\sigma} g^{\mu\rho} g^{\nu\sigma} + j^\mu A_\mu \right)$$

where the factor of $\sqrt{g} \equiv \sqrt{|\det g|}$ is required to make the integration measure coordinate-invariant, and $g^{\mu\nu}$ is the inverse metric: $g^{\mu\nu} g_{\nu\rho} = \delta^\mu_\rho$. Compare the resulting energy-momentum tensor

$$T_g^{\mu\nu} = \frac{2}{\sqrt{g}} \frac{\delta S[A,g]}{\delta g_{\mu\nu}} |_{g_{\mu\nu} = \eta_{\mu\nu}}.$$

with that of the previous part.

Notice that $T_g^{\mu\nu}$ is automatically symmetric and gauge invariant.

[Some useful identities are:

$$\frac{\delta g^{\mu\nu}(x)}{\delta g_{\rho\sigma}(y)} = -g^{\mu\rho} g^{\nu\sigma} \delta^D(x-y) \text{ and } \frac{\delta \det g(x)}{\delta g_{\mu\nu}(y)} = \delta^D(x-y) \det g g^{\mu\nu}.$$

For proofs of these statements see page 93 of this document.]

2. **Maxwell's equations, quantumly.**

(a) Check that the oscillator algebra for the photon creation and annihilation operators

$$[\mathbf{a}_{ks}, \mathbf{a}^\dagger_{k's}] = \delta^3(k-k')\delta_{ss'}. \tag{1}$$

implies (using the mode expansion for $\mathbf{A}$) that

$$[\mathbf{A}_i(\vec{r}), \mathbf{E}_j(\vec{r}')] = -\mathbf{i}\hbar \int đ^3k \; e^{\mathbf{i}\vec{k}\cdot(\vec{r}-\vec{r}')} \left(\delta_{ij} - \hat{k}_i \hat{k}_j \right)$$

(and also $[\mathbf{A}_i(\vec{r}), \mathbf{A}_j(\vec{r}')] = 0$ and $[\mathbf{E}_i(\vec{r}), \mathbf{E}_j(\vec{r}')] = 0$).

Conclude that it's not possible to simultaneously measure $E_x(\vec{r})$ and $B_y(\vec{r})$.

(b) Using the result of the previous part, check that the wave equation for $\mathbf{A}_i(x)$ follows from the Heisenberg equations of motion

$$\partial_t \vec{\mathbf{E}} = \frac{\mathbf{i}}{\hbar} [\mathbf{H}, \vec{\mathbf{E}}].$$

3. **Goldstone boson.** Here is a simple example of the Goldstone phenomenon, which I mentioned briefly in lecture. Consider again the complex scalar field from a previous assignment.

   Suppose the potential is

$$V(\Phi^\star\Phi) = g\left(\Phi^\star\Phi - v^2\right)^2$$

   where g, v are constants. The important features of V are that (1) it is only a function of $|\Phi|^2 = \Phi\Phi^\star$, so that it preserves the particle-number symmetry generated by $\mathbf{q}$ which was the hero a previous homework problem, and (2) the minimum of $V(x)$ away from $x = 0$.

   Treat the system classically. Write the action $S[\Phi, \Phi^\star]$ in polar coordinates in field space:

$$\Phi(x,t) = \rho e^{\mathbf{i}\theta}$$

   where both ρ, θ are functions of space and time.

   (a) Consider constant field configurations, and show that minimizing the potential fixes ρ but not the phase θ.

   (b) Compute the mass^2 of the ρ field about its minimum, $m_\rho^2 = \frac{1}{2}\partial_\rho^2 V|_{\rho=v}$.

   (c) Now ignore the deviations of ρ from its minimum (it's heavy and slow and hard to excite), but continue to treat θ as a field. Plug the resulting expression

$$\Phi = v e^{\mathbf{i}\theta(x,t)}$$

   into the action. Show that θ is a massless scalar field.

   (d) How does the $U(1)$ symmetry generated by $\mathbf{q}$ act on θ?

4. **Casimir force is regulator-independent.** [Bonus problem] Suppose we use a different regulator for the sum in the vacuum energy $\sum_j \hbar\omega_j$. The regulator we'll use here is an analog of Pauli-Villars. We replace

$$f(d) \rightsquigarrow \frac{1}{2}\sum_{j=1}^{\infty} \omega_j K(\omega_j)$$

   where the function K is

$$K(\omega) = \sum_\alpha c_\alpha \frac{\Lambda_\alpha}{\omega + \Lambda_\alpha}.$$

   We impose two conditions on the parameters c_α, Λ_α:

- We want the low-frequency answer to be unmodified:

$$K(\omega) \stackrel{\omega \to 0}{\to} 1$$

– this requires $\sum_\alpha c_\alpha = 1$.

- We want the sum over j to converge; this requires that $K(\omega)$ falls off faster than ω^{-2}. Taylor expanding in the limit $\omega \gg \Lambda_\alpha$, we have

$$K(\omega) \stackrel{\omega \to \infty}{\to} \frac{1}{\omega} \sum_\alpha c_\alpha \Lambda_\alpha - \frac{1}{\omega^2} \sum_\alpha c_\alpha \Lambda_\alpha^2 + \cdots .$$

So we also require $\sum_\alpha c_\alpha \Lambda_\alpha = 0$ and $\sum_\alpha c_\alpha \Lambda_\alpha^2 = 0$.

First, verify the previous claims about $K(\omega)$.

Then compute $f(d)$ and show that with these assumptions, the Casimir force is independent of the parameters c_α, Λ_α.

[A hint for doing the sum: use the identity

$$\frac{1}{X} = \int_0^\infty ds e^{-sX}$$

inside the sum to make it a geometric series. To do the remaining integral over s, Taylor expand the integrand in the regime of interest.]

5. **Casimir energy from balls and springs.** [More difficult bonus problem] Regularize the Casimir energy of a 1d scalar field by discretizing space. If you suppose there are $N \equiv d/a \in \mathbb{Z}$ lattice points in the left cavity

$$| \leftarrow d \rightarrow | \longleftarrow \qquad L - d \qquad \longrightarrow |$$

what answer do you find for the force on the middle plate?

[Hint: you will find the wrong answer! The problem is that with these assumptions d cannot vary continuously. One way to allow d to vary continuously is to impose $\phi(0) = 0 = \phi(d)$, but do not assume d corresponds to a lattice site.]

6. **Gaussian integrals are your friend.**

(a) Show that

$$\int_{-\infty}^\infty dx e^{-\frac{1}{2}ax^2 + jx} = \sqrt{\frac{2\pi}{a}} e^{\frac{j^2}{2a}}.$$

[Hint: square the integral and use polar coordinates.]

(b) Consider a collection of variables $x_i, i = 1..N$ and a real, symmetric matrix a_{ij}. Show that

$$\int \prod_{i=1}^{N} dx_i e^{-\frac{1}{2}x_i a_{ij} x_j + J^i x_i} = \frac{(2\pi)^{N/2}}{\sqrt{\det a}} e^{\frac{1}{2}J^i a_{ij}^{-1} J^j}.$$

(Summation convention in effect, as always.)

[Hint: change integration variables to diagonalize a. $\det a = \prod a_i$, where a_i are the eigenvalues of a.]

(c) I include this problem partly because it might be helpful for a future problem. In that regard, for any function of the N variables, $f(x)$, let

$$\langle f(x) \rangle \equiv \frac{\int \prod_{i=1}^{N} dx_i e^{-\frac{1}{2}x_i a_{ij} x_j} f(x)}{Z[J=0]}, \quad Z[J] = \int \prod_{i=1}^{N} dx_i e^{-\frac{1}{2}x_i a_{ij} x_j + J^i x_i}.$$

Show that

$$\langle x_i x_j \rangle = \partial_{J_i} \partial_{J_j} \log Z[J]|_{J=0} = a_{ij}^{-1}$$

Also, convince yourself that

$$\left\langle e^{J_i x_i} \right\rangle = \frac{Z[J]}{Z[J=0]}.$$

(d) Note that the number N in the previous parts may be infinite. This is really the only path integral we know how to do.

7. **Gaussian identity.** Show that for a gaussian quantum system

$$\left\langle e^{\mathbf{i}K\mathbf{q}} \right\rangle = e^{-A(K)\langle \mathbf{q}^2 \rangle}$$

and determine $A(K)$. Here $\langle ... \rangle \equiv \langle 0| ... |0\rangle$, vacuum expectation value. Here by 'gaussian' I mean that $\mathbf{H}$ contains only quadratic and linear terms in both $\mathbf{q}$ and its conjugate variable $\mathbf{p}$ (but for the formula to be exactly correct as stated you must assume $\mathbf{H}$ contains only terms quadratic in $\mathbf{q}$ and $\mathbf{p}$; for further entertainment fix the formula for the case with linear terms in $\mathbf{H}$).

I recommend using the path integral representation (with hints from the previous problem). Alternatively, you can use the harmonic oscillator operator algebra. Or, even better, do it both ways.