University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215A QFT Fall 2022
Assignment 2

Due 11am Thursday, October 6, 2022

Thanks in advance for following the submission guidelines on hw01. Please ask me by email if you have any trouble.

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1. **Brain-warmer on units.** Briefly show that we can convert between energy, momentum, inverse length, and inverse time by multiplying by various factors of $\hbar$ and c. Thus these dimensionful quantities are all directly comparable once we choose units where $\hbar = c = 1$. What are the length scale and time scale and mass scale associated with 1 GeV $= 10^9$eV? What are the units of the Newton constant of gravity, and what is its value expressed in terms of powers of GeV?

2. **Momentum.** In this problem we consider a scalar field theory in d spatial dimensions. The scalar field is expanded in momentum modes as usual:

$$\phi(\vec{x}) = \int đ^d k \sqrt{\frac{\hbar}{2\omega_k}} \left(e^{\mathbf{i}\vec{k}\cdot\vec{x}} a_k + e^{-\mathbf{i}\vec{k}\cdot\vec{x}} a_k^\dagger \right) .$$

Consider the operator

$$\vec{\mathbf{P}} \equiv \int đ^d k \hbar \vec{k} a_k^\dagger a_k$$

where $\int đ^d k ... \equiv \int \frac{d^d k}{(2\pi)^d} ....$

   (a) Find $[\vec{\mathbf{P}}, a_k^\dagger]$, and $[\vec{\mathbf{P}}, a_k]$.

   (b) Show, using 2a and the mode expansion of the scalar field, that

$$[\vec{\mathbf{P}}, \phi(x)] = \mathbf{i}\hbar\vec{\nabla}\phi(x).$$

   (c) Conclude (using Taylor's theorem) that

$$e^{-\mathbf{i}\vec{a}\cdot\vec{\mathbf{P}}/\hbar} \phi(x) e^{\mathbf{i}\vec{a}\cdot\vec{\mathbf{P}}/\hbar} = \phi(x+a)$$

   and that therefore $\vec{\mathbf{P}}$ generates translations. Therefore $\vec{\mathbf{P}}$ is the operator representing the momentum carried by the field (like the Poynting vector for the electromagnetic field).

   (d) Find $\vec{\mathbf{P}} \left| \vec{k}_1, \vec{k}_2 ... \vec{k}_n \right\rangle$, the action of this operator on a state of n phonons. Conclude that $\hbar\vec{k}$ is the momentum of the phonon labelled by wavenumber $\vec{k}$, as promised in lecture.

3. **Complex scalar field and antiparticles.**

[This problem is related to Peskin problem 2.2.] So far we've discussed scalar field theory with one *real* scalar field. The particles created by such a field are their own antiparticles.

To understand this statement better, consider a scalar field theory in $d+1$ dimensions with *two* real fields ϕ_1, ϕ_2. Organize them into one complex field $\Phi \equiv \frac{1}{\sqrt{2}}(\phi_1 + \mathbf{i}\phi_2)$, with $\Phi^\star = \frac{1}{\sqrt{2}}(\phi_1 - \mathbf{i}\phi_2)$, and let

$$S[\Phi, \Phi^\star] = \int d^d x dt \left(\frac{1}{2}\mu \partial_t \Phi \partial_t \Phi^\star - \frac{1}{2}\mu v^2 \vec{\nabla}\Phi \cdot \vec{\nabla}\Phi^\star - V(\Phi^\star \Phi) \right)$$

for some potential function $V(x)$.

(a) Show that

$$S[\Phi, \Phi^\star] = \int \left(\sum_{i=1,2} \left(A\left(\partial_t \phi_i\right)^2 - B\vec{\nabla}\phi_i \cdot \vec{\nabla}\phi_i \right) - V\left((\phi_1^2 + \phi_2^2)/2 \right) \right) ,$$

and where A, B are constants you must determine. If $V(q^2) = \frac{1}{2}\tilde{m}^2 q^2$, notice that the action is just the sum of two copies of the action of the theory we considered previously.

(b) Show by doing the Legendre transformation that the associated hamiltonian is

$$\mathbf{H} = \int d^d x \left(C\Pi\Pi^\star + D\vec{\nabla}\Phi \cdot \vec{\nabla}\Phi^\star + V\left(\Phi\Phi^\star\right) \right)$$

where C, D are constants you must determine, and the canonical momenta are

$$\Pi = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}} = \frac{1}{2}\mu \dot{\Phi}^\star, \quad \Pi^\star = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}^\star} = \frac{1}{2}\mu \dot{\Phi}$$

with the Lagrangian density $\mathcal{L}$ defined by $S = \int dt d^d x \mathcal{L}$.

(c) This theory has a continuous symmetry under which $\Phi \to e^{\mathbf{i}\alpha}\Phi, \Phi^\star \to e^{-\mathbf{i}\alpha}\Phi^\star$ with α a real constant. Show that the action S does not change if I make this replacement. [1]

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[1] This is called a U(1) symmetry: it is a unitary rotation (hence 'U') on a one-dimensional (hence '(1)') complex vector. Notice that on the real components ϕ_1, ϕ_2 it acts as a two-dimensional rotation:

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \to \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} .$$

The name for this group is SO(2). So U(1) is the same as SO(2).

(d) The existence of a continuous symmetry means a conserved charge – a hermitian operator that commutes with the Hamiltonian, which generates the symmetry (this is the quantum version of the Noether theorem). Show that

$$\mathbf{q} \equiv \int d^d x \, \mathbf{i} \left(\Phi^\star \Pi^\star - \Pi \Phi\right)$$

generates this transformation, in the sense that the change in the field under a transformation with infinitesimal α is

$$\delta\Phi = \mathbf{i}\alpha\Phi = -\mathbf{i}\alpha[\mathbf{q}, \Phi], \quad \text{and} \quad \delta\Phi^\star = -\mathbf{i}\alpha\Phi^\star = -\mathbf{i}\alpha[\mathbf{q}, \Phi^\star].$$

Show that $[\mathbf{q}, \mathbf{H}] = 0$.

(e) For the case where $V(\Phi\Phi^\star) = \frac{1}{2}\tilde{m}^2\Phi\Phi^\star$ the hamiltonian is quadratic. Diagonalize it in terms of *two sets* of creation operators and annihilation operators. Work in the continuum. You should find something of the form

$$\Phi = \sqrt{\frac{\hbar}{2\mu}} \sum_k \frac{1}{\sqrt{\omega_k}} \left(e^{\mathbf{i}kx} \mathbf{a}_k + e^{-\mathbf{i}kx} \mathbf{b}_k^\dagger \right). \qquad (1)$$

(f) Write the canonical commutators

$$[\Phi(x), \Pi(x')] = \mathbf{i}\hbar\delta(x - x'), \qquad [\Phi(x), \Pi^\star(x')] = 0$$

(and the hermitian conjugate expressions) in terms of $\mathbf{a}$ and $\mathbf{b}$.

(g) Rewrite $\mathbf{q}$ in terms of the mode operators.

(h) Evaluate the charge of each type of particle created by $\mathbf{a}_k^\dagger$ and $\mathbf{b}_k^\dagger$ (*i.e.* find $[\mathbf{q}, \mathbf{a}^\dagger]$).

I claim that the particle created by $\mathbf{a}^\dagger$ is the antiparticle of that created by $\mathbf{b}^\dagger$ in the sense that they have opposite quantum numbers. This means that we can add terms to the hamiltonian by which they can annihilate each other, without breaking any symmetries. What might such a term look like?

The following problem is postponed until the next problem set.

4. **Classical Maxwell theory.** [Peskin problem 2.1, lightly edited] Classical electromagnetism follows from the action

$$S[A] = \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^\mu A_\mu \right), \quad \text{where } F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

(a) Derive Maxwell's equations as the Euler-Lagrange equations of this action, treating the components $A_\mu(x)$ as the dynamical variables

$$0 = \frac{\delta S[A]}{\delta A_\mu(x)}.$$

Write the equations in the standard form by identifying $E^i = -F^{0i}$ and $\epsilon^{ijk} B^k = -F^{ij}$.

(b) Construct the energy-momentum tensor for this theory, when $j^\mu = 0$. Note that the usual procedure

$$T^\mu_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \mathcal{L} \delta^\mu_\nu$$

does not result in a symmetric tensor. (It is also not gauge invariant.) To remedy that, we can add to $T^{\mu\nu}$ a term of the form $\partial_\lambda K^{\lambda\mu\nu}$, where $K^{\lambda\mu\nu}$ is antisymmetric in its first two indices. Such an object is automatically divergenceless, so

$$\widehat{T}^{\mu\nu} \equiv T^{\mu\nu} + \partial_\lambda K^{\lambda\mu\nu}$$

is an equally good energy-momentum tensor with the same globally conserved energy and momentum. Show that this construction, with

$$K^{\lambda\mu\nu} = F^{\mu\lambda} A^\nu,$$

leads to an energy-momentum tensor $\widehat{T}$ that is symmetric and yields the standard formulae for the electromagnetic energy and momentum densities:

$$\mathcal{E} = \frac{1}{2}\left(E^2 + B^2\right), \quad \vec{S} = \vec{E} \times \vec{B}.$$

(c) [Bonus problem] A better way to think about the energy-momentum tensor is to regard it as the response to a change in the background metric. (This is why it appears as a source in Einstein's equations.) To couple the Maxwell theory to a general background metric $g_{\mu\nu}$, we replace all the $\eta_{\mu\nu}$s with $g_{\mu\nu}$s:

$$S[A, g] = \int d^4x \sqrt{g} \left(-\frac{1}{4} F_{\mu\nu} F_{\rho\sigma} g^{\mu\rho} g^{\nu\sigma} + j^\mu A_\mu \right)$$

where the factor of $\sqrt{g} \equiv \sqrt{|\det g|}$ is required to make the integration measure coordinate-invariant, and $g^{\mu\nu}$ is the inverse metric: $g^{\mu\nu} g_{\nu\rho} = \delta^\mu_\rho$. Compare the resulting energy-momentum tensor

$$T_g^{\mu\nu} = \frac{2}{\sqrt{g}} \frac{\delta S[A, g]}{\delta g_{\mu\nu}} |_{g_{\mu\nu} = \eta_{\mu\nu}}.$$

with that of the previous part.

Notice that $T_g^{\mu\nu}$ is automatically symmetric and gauge invariant.

[Some useful identities are:

$$\frac{\delta g^{\mu\nu}(x)}{\delta g_{\rho\sigma}(y)} = -g^{\mu\rho} g^{\nu\sigma} \delta^D(x-y) \text{ and } \frac{\delta \det g(x)}{\delta g_{\mu\nu}(y)} = \delta^D(x-y) \det g g^{\mu\nu}.$$

For proofs of these statements see page 93 of this document.]