6.3 Vector Fields

Most general $\mathcal{L}$ for A_μ w/

- Lorentz
- at most 2 derivs
- 2 A_μ's

$$\mathcal{L} = -\frac{1}{2}\Big(\partial_\mu A^\nu \partial^\mu A_\nu + a \underbrace{\partial_\mu A^\mu \partial_\nu A^\nu}_{=(\partial\cdot A)^2} + b A_\mu A^\mu$$

$$+ c \underbrace{\epsilon^{\mu\nu\rho\sigma}\partial_\mu A_\nu \partial_\rho A_\sigma}_{\propto \partial_\mu\left(\epsilon^{\mu\nu\rho\sigma} A_\nu \partial_\rho A_\sigma\right)}\Big)$$

"θ term"

invisible in pert. thy.

$$0 = \frac{\delta}{\delta A^\nu(x)}\int \mathcal{L} =$$

$$-\partial^2 A_\nu - a\partial_\nu(\partial\cdot A) + bA_\nu$$

$$A_\mu(x) = \epsilon_\mu e^{-ik\cdot x}$$

$$\Rightarrow \quad k^2\epsilon_\mu + a k_\mu(\epsilon\cdot k) + b\epsilon_\mu = 0.$$

If $\epsilon\cdot k \neq 0 \Rightarrow$ $\underline{\underline{k^2 = -\frac{b}{1+a}}} \rightarrow \infty$ if $a \rightarrow -1$ w/ $b \neq 0$.

If $\epsilon\cdot k = 0 \Rightarrow k^2 = -b.$

$$\mathcal{L}_{a=-1,\, b=-\mu^2} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\mu^2 A_\mu A^\mu$$

(Proca)

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$$

<u>eom</u>: $\quad 0 = \partial^\mu F_{\mu\nu} + \mu^2 A_\nu$

$\underset{\partial^\nu(\text{BHS})}{\Longrightarrow} \quad 0 = \partial^\mu \partial^\nu F_{\mu\nu} + \mu^2 \partial^\nu A_\nu$

$\partial^\mu \partial^\nu F_{\mu\nu} \to 0$ (Bianchi id) $\quad \Rightarrow \partial \cdot A = 0$

$\Longrightarrow \quad -\partial^2 A_\nu + \mu^2 A_\nu = 0.$ KG for each component.

$$\boxed{\begin{array}{l} \text{ie} \quad k^2 = \mu^2. \\ \text{and} \quad \epsilon \cdot k = 0. \end{array}}$$

In the rest frame: $k^\mu = (k^0, 0)^\mu$

$$\epsilon^{(\pm 1)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ \pm i \\ 0 \end{pmatrix} \qquad \epsilon^{(0)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

are evecs of

$$J^z \doteq i \begin{pmatrix} 0 & & \\ & & +1 \\ & -1 & \end{pmatrix}$$

w/ evals ± 1 and 0.

when $\mu \to 0$ $\quad k^\mu = (\omega, 0, 0, \omega)$
and only two satisfy $\epsilon \cdot k = 0$.

$\epsilon^{\pm,0}$ satisfy : $\quad \epsilon^{(r)} \cdot \epsilon^{(s)} = -\delta^{rs}$

and
$$\sum_{r=0,\pm 1} \epsilon_\mu^{(r)*} \epsilon_\nu^{(r)} = -\eta_{\mu\nu} - \frac{k_\mu k_\nu}{\mu^2}.$$

Away from rest frame, say $k^\mu = (\omega, 0, 0, p^z)$

then $\epsilon^{(0)}_\mu = \begin{pmatrix} p_z/\mu \\ 0 \\ 0 \\ -\omega/\mu \end{pmatrix}$. (longitudinal polarization)

Canonical Quantization !
$$\pi^i = \frac{\partial \mathcal{L}}{\partial \dot{A}_i} = -F^{0i} = E^i.$$
$$\pi^0 = \frac{\partial \mathcal{L}}{\partial \dot{A}_0} = 0.$$

i.e. A_0 is an algebraic variable

$$Z = \int DA_0 D\vec{A} \; e^{iS[\vec{A}, \partial_\mu \vec{A}, \underline{A_0}]}$$

A_0 is a Lagrange multiplier Setting $0 = \frac{\delta S}{\delta A_0} = 0$

$$0 = \frac{\delta S}{\delta A_0} = \vec{\nabla}\cdot\vec{E} - \mu^2 A_0 \quad - 4\pi\rho$$

$$= (-\nabla^2 + \mu^2) A_0 + \vec{\nabla}\cdot\dot{\vec{A}}$$

$$\Longrightarrow \quad A_0(\vec{x},t) = \underbrace{\left(\frac{1}{-\nabla^2+\mu^2}\right)}\left(-\vec{\nabla}\cdot\dot{\vec{A}}\right)(\vec{x},t)$$

$$\boxed{A_0(t) \text{ is determined by } \vec{A}(t)} \qquad = \int d^3y \, e^{-\mu|\vec{x}-\vec{y}|} \frac{(-\vec{\nabla}\cdot\dot{\vec{A}})(y,t)}{4\pi|\vec{x}-\vec{y}|}$$

$$h = \frac{1}{2}\left(\vec{E}^2 + \vec{B}^2 + \mu^2\vec{A}^2 + \mu^2 A_0^2\right) \geq 0.$$

Canonical ETCR:

$$[A_i(\vec{x},t), F^{j0}(\vec{y},t)] = i\delta_i^j \, \delta^{(3)}(\vec{x}-\vec{y})$$

$$A_\mu(x) = \sum_{r=0,\pm1} \int \frac{d^3k}{\sqrt{2\omega_k}} \left(e^{-ikx} a_k^r \epsilon_\mu^r + e^{+ikx} a_k^{r+} \epsilon_\mu^{r*}\right)$$

$(A = A^+.)$

$$\Longrightarrow \quad [a_k^r, a_p^{s\dagger}] = \delta^{(3)}(\vec{k}-\vec{p})\,\delta^{rs}$$

$$\Longrightarrow \quad :H: = \sum_r \int d^3k \; a_k^{\dagger r} a_k^r \,\omega_k \,.$$

plug in mode expansion

$$\langle 0| T A_\mu(x) A_\nu(y) |0\rangle = \left[\int d^4k \; e^{-ik(x-y)} \left[\frac{-i\left(\eta_{\mu\nu} - k_\mu k_\nu/\mu^2\right)}{k^2-\mu^2+i\epsilon} \right] \right]$$

$$\sum_r \epsilon_\mu^r \epsilon_\nu^{*r}$$

$$\left[\langle A_i A_j \rangle \sim \frac{-i\eta_{ij}}{k^2+\dots} = \frac{-i(-\delta_{ij})}{k^2+\dots} \right.$$

like 3 scalars. $\Big]$

$$\left.\begin{aligned} \langle 0| A_\mu(x) | k, r\rangle &= \epsilon_\mu^r(k)\, e^{-ikx} \\ \langle k, r| A_\mu(x) |0\rangle &= \epsilon_\mu^r(k)^* e^{+ikx} \end{aligned}\right\}$$

Massless Case : Consider coupling to a current :

$$\Delta \mathcal{L} = A_\mu j^\mu$$

$$\xrightarrow{\text{Bianch}} \partial_\mu A^\mu = \mu^{-2} \partial_\mu j^\mu$$

BAD as $\mu \to 0$ unless $\underline{\underline{\partial_\mu j^\mu = 0}}$.

<u>eg</u>: [QED] $j^\mu = eq \bar{\Psi} \gamma^\mu \Psi$

$A_\mu j^\mu$ comes "minimal coupling"

$$\partial_\mu \to D_\mu = \partial_\mu + ieqA_\mu$$

in the Dirac Lagrangian.

The resulting $\mathcal{L}$ has AN INVARIANCE :

$$\begin{cases} A_\mu(x) \to A_\mu + \partial_\mu \lambda / e \\ \Psi(x) \to e^{-iq\lambda(x)} \Psi \end{cases} \quad \forall \lambda(x).$$

Gauge invariance IS NOT A SYMMETRY but a REDUNDANCY.

- They have the same $\vec{E}, \vec{B}$, $\oint_C A$.

- If not, the kinetic operator K
$$S = \frac{1}{2}\int A \underline{K} A$$
would not be invertible.

$$(\underline{K} A)_\mu = \left(\eta_{\mu\nu}\partial^\rho\partial_\rho - \partial_\mu\partial_\nu\right)A^\nu$$

<u>Since:</u> $KA\Big|_{A_\mu = \partial_\mu\lambda} = 0$.

$$Z = \underline{\int \mathcal{D}A}\; e^{-\frac{1}{2}\int AKA} = \sqrt{\frac{\pi^\#}{\det K}}$$

$\Rightarrow \epsilon_\mu \propto k_\mu$ is $A_\mu = \partial_\mu \lambda$ gauge-trivial.

eg 2 : Scalar QED

$$\mathcal{L}_0 = \partial_\mu \Phi \, \partial^\mu \Phi^* - V(|\Phi|^2)$$

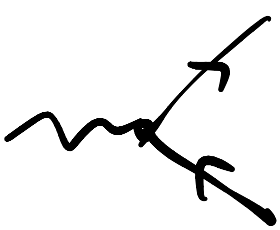
$\mathcal{L}_{\text{Maxwell}}$

$$\partial_\mu \to D_\mu$$

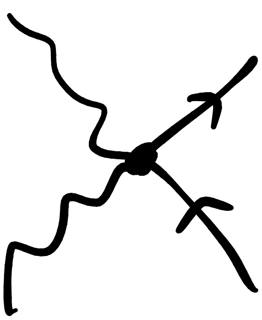
$$\rightsquigarrow \mathcal{L}_{\substack{\text{scalar}\\ \text{QED}}} = D_\mu \Phi \, D^\mu \Phi^* + \dots$$

$$= |\partial \Phi|^2 + \# A_\mu \Phi \, \underline{\underline{\partial^\mu \Phi^*}}$$

$$+ A_\mu A^\mu |\Phi|^2 + \dots$$



$$= -ieq(p_\Phi + p_{\Phi^*})^\mu$$



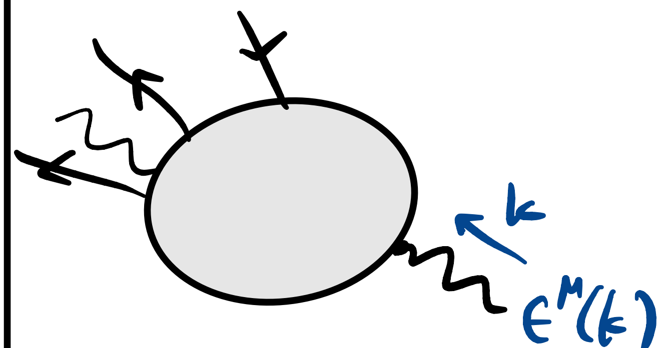
$$= -ie^2 q^2 \eta_{\mu\nu}$$

What's the propagator: Strategy 1: use the one w/ μ & take $\mu \to 0$ at the end.

Claim: the terms w/ $\frac{k_\mu k_\nu}{\mu^2}$ never contribute.

because of gauge invariance.

Ward identity:



$$= i\mathcal{M} \equiv \underbrace{i\mathcal{M}^\mu(k)}\, \underline{\epsilon_\mu(k)}$$

then if all ext. lines are on-shell

$$\mathcal{M}^\mu(k)\, k_\mu = 0.$$

why? $k_\mu \mathcal{M}^\mu \sim \lim_{\cdots} (\square - m^2) \cdots \langle \Omega | \cdots k_\mu j^\mu(k) \cdots | \Omega \rangle$

$\to 0.$

$$j^\mu(k) \equiv \int d^4x\, e^{-ikx} j^\mu(x)$$

$$\partial_\mu j^\mu(x) \implies k_\mu j^\mu(k) = 0$$

$\Longrightarrow$ we never make longitudinal photons:

$$\mathcal{A}\left(\begin{matrix}\text{emit } \epsilon_L^{\lambda} = (k,0,0,-\omega)^{\lambda}/\mu \\ \text{w } k^{\lambda} = (\omega,0,0,k)^{\lambda}\end{matrix}\right)$$

$$\left[\begin{matrix}\epsilon_L^{\lambda} k_{\lambda} = 0) \\ \text{and} \\ \epsilon_L^{\lambda} \cdot \epsilon_{L\lambda} = -1.\end{matrix}\right.$$

$$\propto \quad \epsilon_{\mu}^{L} \mathcal{M}^{\mu}$$

$$= \frac{1}{\mu}\left(k\mathcal{M}^0 - \omega\mathcal{M}^3\right)$$

$$= \frac{1}{\mu}\Big(k\mathcal{M}^0 - \underbrace{\sqrt{k^2+\mu^2}}_{= k + \frac{\mu^2}{2k} + \cdots}\mathcal{M}^3 \Big)$$

$$= \frac{1}{\mu} \underbrace{k_{\mu}\mathcal{M}^{\mu}}_{\to 0 \text{ by Ward id.}} - \underbrace{\frac{\mu}{2k}\mathcal{M}^3 + O(\mu^3)}_{\to 0 \text{ when } \mu \to 0}.$$

Gauge-Fixing:

eg 1 Coulomb gauge: $\partial_\mu A^\mu = 0$

AND $\vec{\nabla}\cdot\vec{A} = 0$.

$$A_0 = \int_y G(y,x)\left[\cancel{\nabla\cdot\dot{\vec{A}}} + 4\pi\rho\right]_y$$

$= 0$ Coulomb gauge

$\vec{\nabla}\cdot\vec{A} \propto \epsilon^{(0)} = 0$.

Removes longitudinal mode.

PRICE: Lorentz inv X

Coulomb force is instantaneous.

eg 2 "R_ξ-gauges": discourage $\partial\cdot A \neq 0$:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial\cdot A)^2$$

w/ the $\frac{1}{2\xi}$ term K is invertible:

$$\langle T A_\mu(x) A_\nu(y) \rangle =$$

$$\int d^4k \; e^{-ik(x-y)} \left[\frac{-i \left(\eta_{\mu\nu} - (1-\xi) k_\mu k_\nu / k^2\right)}{k^2 - \mu^2 + i\epsilon} \right) \checkmark$$

$\xi = 1$ "Feynman gauge".

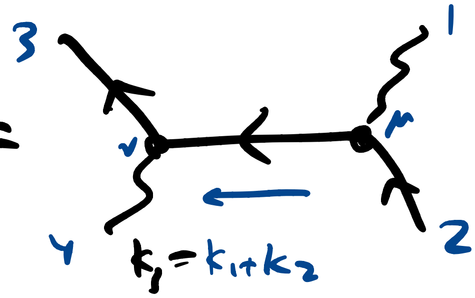
$\xi = 0$ "Landau gauge" <u>projector onto $k_\perp$</u>.

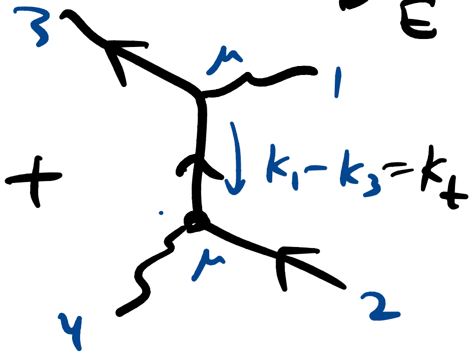
$$\Pi_{\mu\nu} = \eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}$$

satisfies
$$\begin{cases} \Pi_{\mu\nu} \Pi^\nu{}_\rho = \Pi_{\mu\rho} \\ \Pi_{\mu\nu} k^\nu = 0. \end{cases}$$

More examples:

$e^- - \gamma$ scattering

$i\mathcal{M}_{e\gamma\leftarrow e\gamma} =$ 

$+$ 

$$= i\mathcal{M}_s + i\mathcal{M}_t$$

$$= (-ie)^2 \epsilon_1^{\mu} \epsilon_4^{\nu *} \bar{u}_3 \left[\gamma_\nu \frac{i\not{k}_s + m}{s - m^2} \gamma_\mu \right.$$

$$\left. + \gamma_\mu \frac{i\not{k}_t + m}{t - m^2} \gamma_\nu \right] u_2$$

unpolarized scattering

$$P = \frac{1}{4} \sum_{\substack{\text{pols} \\ \text{spins}}} |\mathcal{M}|^2$$

$$(*)\ \sum_{r=1,2} e^{r\,*}_\mu(k)\, e^r_\nu(k) = -\eta_{\mu\nu} + k_\mu k_\nu$$

DOES NOT MATTER

$$i\mathcal{M}_{e^-\gamma \leftarrow e^-\gamma} = \epsilon_1^{\mu}\, \mathcal{M}_\mu$$

$$\rightsquigarrow\ k^\mu \mathcal{M}_\mu = 0.$$

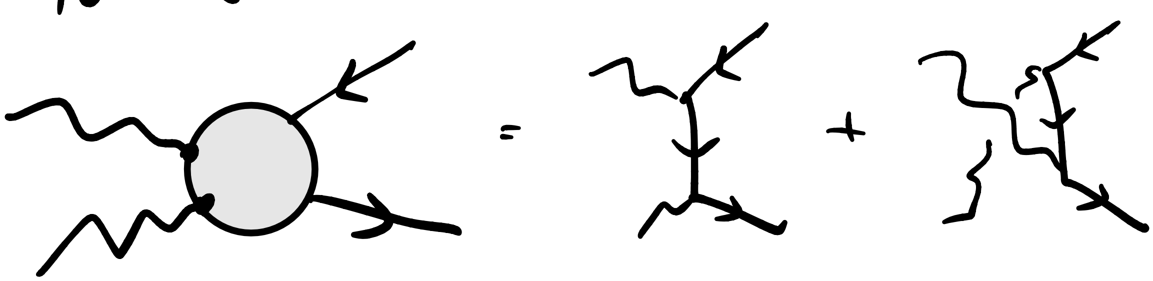
$$\sum_{\text{pols}} |\mathcal{M}|^2 = \sum_r \epsilon^{r*}_\mu\, \mathcal{M}^{\mu *} \mathcal{M}^\nu \epsilon^r_\nu$$

$$\overset{(*)}{=} -\eta_{\mu\nu}\mathcal{M}^\mu \mathcal{M}^\nu + \cancel{\#\mathcal{M}^\mu k_\mu}^{=0} \cdots$$

$$= -\mathcal{M}_\mu \mathcal{M}^\mu. \qquad \underline{\geq 0}.$$

__eg:__ $e^-\gamma \leftarrow e^-\gamma$ is related by cossing

to $\gamma\gamma \leftarrow e^+e^-$



$$\phi(x) \longmapsto \phi'(x) = \phi(\Lambda^{-1}x)$$

$$\Psi(x) \longrightarrow e^{-i\theta_{\mu\nu}J^{\mu\nu}}\, \Psi(x)\, e^{+i\theta_{\mu\nu}J^{\mu\nu}}$$

part a: $\Psi \rightarrow e^{-i\alpha Q}\, \Psi\, e^{+i\alpha Q}$

$$= e^{-i\alpha}\, \Psi$$

$$\Lambda_{\frac{1}{2}}(\theta,\beta) = e^{-i\,(\underbrace{\theta\cdot J_{\frac{1}{2}} + \beta\cdot K_{\frac{1}{2}}}_{\theta_{\mu\nu}J^{\mu\nu}_{Dirac}})}$$

Dirac Rep.

$\theta_{\mu\nu}J^{\mu\nu}_{Dirac}$

$\sigma^{\mu\nu} \propto [\gamma^\mu, \gamma^\nu]$.

$$\bar{u}(p')\gamma^\mu u(p) = \bar{u}(p')\left[\frac{(p+p')^\mu + \sigma^{\mu\nu}(p-p')_\nu}{2m}\right] u(p)$$