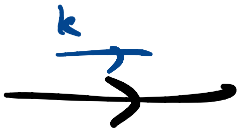
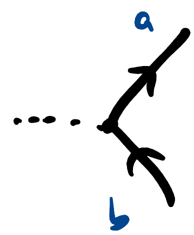
Feynman rules w/ fermions, cont'd

eg: $\mathcal{L}_{Yukawa} = \bar\Psi(i\not\partial - m_\psi)\Psi + \frac{1}{2}\left((\partial\phi)^2 - m_\phi^2\phi^2\right)$
$\qquad + \mathcal{L}_{int}$

$$\mathcal{L}_{int} = -g\,\phi\,\bar\Psi\Psi \,.$$



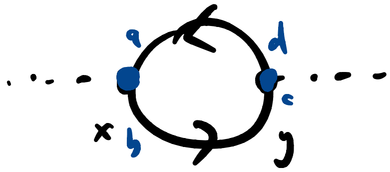
$= \dfrac{i}{\not k - m_\psi}$



$= -ig\,\delta^{ab}.$

$= \dfrac{i}{k^2 - m_\phi^2}$

Rule 8: A diagram w/ L fermion loops gets a $(-1)^L$.

eg: 

δ_{ab} $\qquad\qquad\qquad\qquad$ δ_{cd}

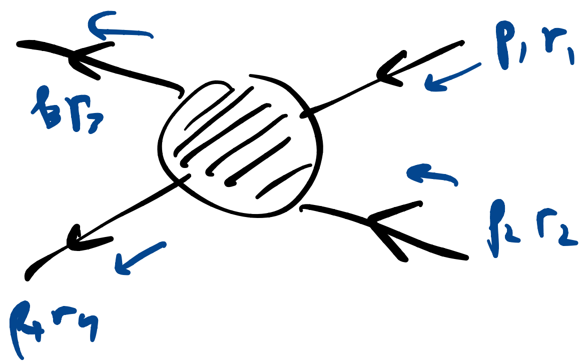
$$\propto \sum_{abcd} \bar\Psi_a(x)\Psi_b(x)\,\bar\Psi_c(y)\Psi_d(y)$$

$$= (-1)^3\, \Psi_b(x)\bar\Psi_c(y)\; \Psi_d(y)\bar\Psi_a(x)\; \delta^{ab}\delta^{cd}$$

$$= (-1)\,\mathrm{tr}\, S_F(x-y)\, S_F(y-x) \,.$$

Rule 7: Diagrams related by exchanging external fermi lines get a (-1).

eg: $\mathcal{M}_{\Psi\Psi \leftarrow \Psi\Psi}$ (nucleon scattering)



$$\propto {}_0\langle p_3 r_3 ; p_4 r_4 | T e^{-i\int \mathcal{L}_{int} d^4 z} | p_1 r_1 ; p_2 r_2 \rangle_0$$

up to $-i\delta^4(\Sigma p)$

where: $|p_1 r_1 ; p_2 r_2\rangle_0 \propto a_{p_1}^{r_1 \dagger} a_{p_2}^{r_2 \dagger} |0\rangle$

$${}_0\langle p_3 r_3 ; p_4 r_4 | = \left(| p_3 r_3 ; p_4 r_4 \rangle \right)^\dagger$$

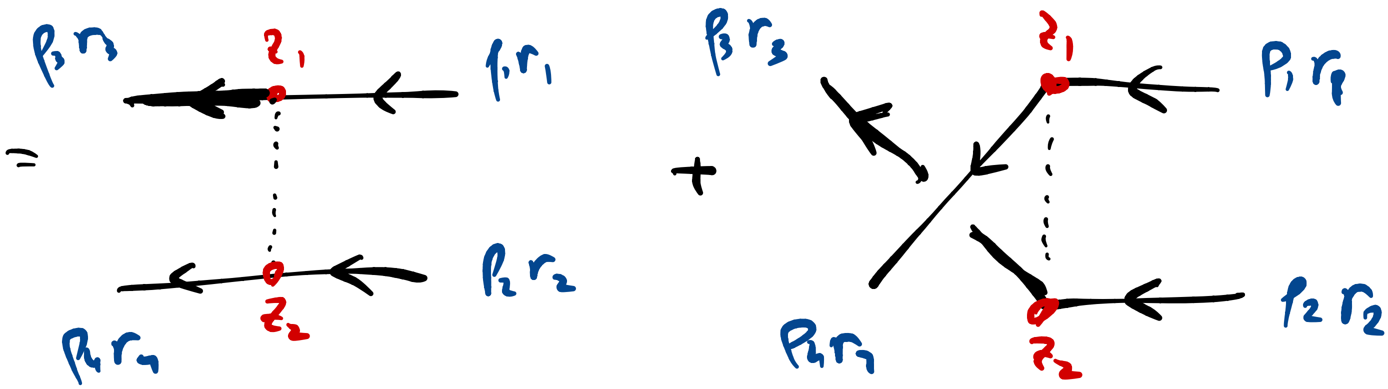
$$= \langle 0 | a_{p_4}^{r_4} a_{p_3}^{r_3}$$

$$= -\langle 0 | a_{p_3}^{r_3} a_{p_4}^{r_4}$$

$$S_{fi} = \langle p_3 r_3 ; p_4 r_4 | T \left(\frac{1}{2!}\right) (-ig)^2 \int d^D z_1 \int d^D z_2$$

$$(\bar{\Psi}\Psi\phi)_1 (\bar{\Psi}\Psi\phi)_2 \; | p_1 r_1 ; p_2 r_2 \rangle$$

$$+ O(g^4)$$



$+ (z_1 \leftrightarrow z_2)$ $\qquad$ $+ (z_1 \leftrightarrow z_2)$

<u>convention:</u>

$$\langle p_3 p_4 | (\mathbb{1} + \dots) | p_1 p_2 \rangle = + \delta(p_1 - p_3)\, \delta(p_2 - p_4)$$

$$\langle 0 | a_{p_4} a_{p_3} a_1^\dagger a_2^\dagger \dots + \langle 0 | a_{p_4} a_{p_3} a_1^\dagger a_2^\dagger \dots$$

$$= \langle 0 | a_{p_3} a_1^\dagger \; a_{p_4} a_2^\dagger \dots - \langle 0 | a_{p_4} a_1^\dagger \; a_{p_3} a_2^\dagger \dots$$

$$|\mathcal{M}_1 - \mathcal{M}_2|^2 = |\mathcal{M}_1|^2 + |\mathcal{M}_2|^2 - \mathcal{M}_1 \mathcal{M}_2^* - \mathcal{M}_2 \mathcal{M}_1^*$$

$$S_{fi} = -g^2 \int d^4 z_1 \int d^4 z_2 \int d^4 q \, e^{-iq(z_1 - z_2)} \frac{i}{q^2 - m_\phi^2 + i\epsilon}$$

$$\times \, e^{-i z_1 (p_1 - p_3)} \bar{u}^{r_3}(p_3) u^{r_1}(p_1)$$

$$\times \, e^{-i z_2 (p_2 - p_4)} \bar{u}^{r_4}(p_4) u^{r_2}(p_2)$$ ← t channel

$$- \left(3 \longleftrightarrow 4 \right)$$ ↙ u channel

$$\int_q \int_{z_1} \int_{z_2} \xrightarrow{t} \int_q \delta(p_1 - p_3 - q)\, \delta(p_2 - p_4 - q)$$

$$= \delta(p_1 + p_2 - p_3 - p_4)$$

$$\xrightarrow{u} \int \delta(p_1 - p_4 - q)\, \delta(p_2 - p_3 - q) = \delta\left(\sum p_i\right)$$

$$(S - \mathbb{1})_{fi} = \delta^4(p_T)\, i\mathcal{M}$$

$$\Rightarrow i\mathcal{M} = -ig^2\Big(\frac{1}{t - m_\phi^2} (\bar{u}_3 u_1)(\bar{u}_4 u_2)$$

$$- \frac{1}{u - m_\phi^2} (\bar{u}_4 u_1)(\bar{u}_3 u_2) \Big)$$

(fermi statistics)

$$t = (p_1 - p_3)^2$$

$$u = (p_1 - p_4)^2 .$$

for each $\vec{p}$ $\quad \underline{u^r(\vec{p})} \quad \underline{v^s(\vec{p})}$

$r, s = 1, 2$.

<u>Yukawa force</u>: NR limit : $p_1^\mu \simeq (m, \vec{p})^\mu$

COM frame : $p_2^\mu \simeq (m, -\vec{p})^\mu$

$$p_3^\mu \simeq (m, \vec{p}\,')^\mu$$

$$p_4^\mu \simeq (m, -\vec{p}\,')^\mu .$$

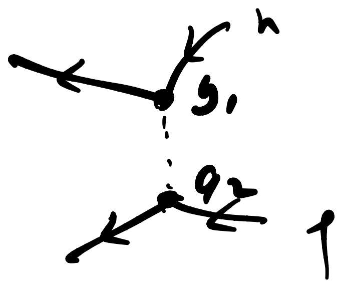
$$u^r(\vec{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\, \xi^r \\ \sqrt{p\cdot\bar{\sigma}}\, \xi^r \end{pmatrix} \xrightarrow{NR} \sqrt{m} \begin{pmatrix} \xi^r \\ \xi^r \end{pmatrix}$$

$$\rightarrow \bar{u}_3 u_1 \equiv \bar{u}(p_3)^{r_3}\, u(p_1)^{r_1}$$

$$= 2m \underbrace{\xi^{\dagger}_{r_3}\, \xi_{r_1}}_{} = 2m\, \delta^{r_3 r_1}.$$

Scatter distinguishable ferms:

$$\mathcal{L}_{int} = g_1 \bar{\Psi}_n \Psi_n \phi + g_2 \bar{\Psi}_p \Psi_p \phi .$$



$$q = p_1 - p_3 = (0, \vec{p} - \vec{p}\,')$$

$$\rightarrow t = (p_1 - p_3)^2$$

$$= -\vec{q}^{\,2} = -(\vec{p} - \vec{p}\,')^2 < 0.$$

$$\rightarrow i\mathcal{M}_{NR,COM} = i\, g_1 g_2 \frac{i}{\vec{q}^{\,2} + m_\phi^2}\, 4m^2 \delta^{r_1 r_3} \delta^{r_2 r_4}$$

[Spin is preserved.]

Compare to NR QM (Born):

$$\delta^{r_1 r_3}\, 2\pi \delta(E_{\vec{p}} - E_{\vec{p}'}) \left(-i \tilde{U}(\vec{q})\right) = {}_{NR}\langle \vec{p}\,', r_3 | \hat{S} | \vec{p}, r_1 \rangle_{NR}$$

$$\mathcal{X} = \underbrace{\sum_{r_4}\int d^3p_4}_{\left(\sum \text{final states}\right)} \prod_{i=1}^{4}\frac{1}{\sqrt{2E_i}}\;\underbrace{S(34 \leftarrow 12)}_{= i\mathcal{M}_{NRcom}\,\delta^4(\Sigma p_T)}$$

$$= 2\pi\,\delta(E_p - E_{p'})\,\delta^{r_1 r_3}\;\underbrace{\frac{-g_1 g_2}{\vec{q}^{\,2} + m_\phi^2}}_{= -i\tilde{U}(\vec{q})}.$$

$$\Longrightarrow \boxed{U(x) = -\frac{g_1 g_2\, e^{-m_\phi r}}{4\pi r}}$$

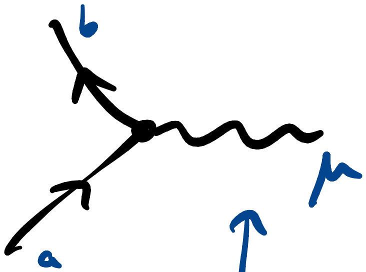
(attractive if $g_1 g_2 > 0$.)

Chapter 6 : QED

$$\mathcal{L} = \mathcal{L}_{Dirac} + \mathcal{L}_{photon} + \mathcal{L}_{int}$$

$$\mathcal{L}_{int} = -e\, \underline{\bar{\Psi} \gamma^\mu \Psi A_\mu} = \underline{j}^\mu A_\mu .$$

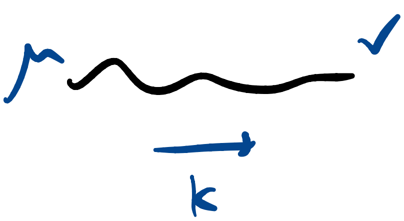
9.



$$= -ie\,\gamma^\mu_{ba}$$

nubbin to which a photon line attaches

10. An internal photon line gets



$$= \frac{i}{k^2 - m_\gamma^2 + i\epsilon}\left(-\eta^{\mu\nu} + (1-\xi)\frac{k^\mu k^\nu}{k^2}\right)$$

where $m_\gamma = 0$ and ξ is arbitrary.

11. External photon

in the initial state gets $\epsilon^{\mu}(\vec{p})$

" " final " " $\epsilon^{\mu *}(\vec{p})$.

6.3 QED processes at leading order

in $\alpha = \frac{e^2}{4\pi} \simeq \frac{1}{137}$

$$j^\mu = \bar{\Psi}\gamma^\mu \Psi .$$

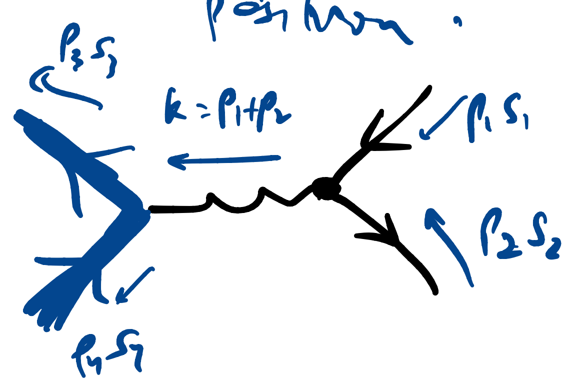
$$\int d^3x \, j^0(x) = \int \bar{\Psi}\gamma^0\Psi = \int_x \Psi_x^\dagger \Psi_x$$

$$= \int d^3p \sum_s \Big(\underbrace{a^\dagger_{\vec{p}s} a_{\vec{p}s}}_{\#\text{ of } e^-\text{s}} - \underbrace{b^\dagger_{\vec{p}s} b_{\vec{p}s}}_{\#\text{ of positrons}} \Big)$$

$a^\dagger$: creates an electron

$b^\dagger$: creates a positron.

e^1: $\quad i\mathcal{M}_{\mu^+\mu^- \leftarrow e^+e^-} =$



$$= \left(-ie\, \bar{u}^{s_3}(\vec{p}_3)\, \gamma^\mu\, v^{s_4}(\vec{p}_4) \right)_{\text{muons}}$$

$$\times \frac{-i\left(\eta_{\mu\nu} - \frac{(1-\xi)k_\mu k_\nu}{k^2}\right)}{k^2}$$

ie $\not{p}_3 u_3 = m_\mu u_3$

$$\left(-ie\, \bar{v}^{s_2}(\vec{p}_2)\, \gamma^\nu\, u^{s_1}(p_1) \right)_{\text{electron}}$$

$\not{p}_1 u_1 = m_e u_1$

$$k = p_1 + p_2 = p_3 + p_4 .$$

<u>Illustration</u> of "<u>Ward identity</u>" (ξ doesn't matter):

$\underline{\not{p}_1 u_1 = m_e u_1}$ $\quad \underline{\bar{v}_2 \not{p}_2 = -m_e \bar{v}_2}$.

$$k_\nu \bar{v}_2 \gamma^\nu u_1 \overset{k = p_1 + p_2}{=} \bar{v}_2 (\not{p}_1 + \not{p}_2) u_1$$

$$= \bar{v}_2 \underbrace{\not{p}_1 u_1}_{m_e u_1} + \underbrace{\bar{v}_2 \not{p}_2}_{-m_e \bar{v}_2} u_1 = 0 .$$

$$\Rightarrow \mathcal{M} = \frac{e^2}{s} (\bar{u}_3 \gamma_\mu v_4)_{\text{muons}} (\bar{v}_2 \gamma^\mu u_1)_{e^-}$$

$$s = (p_1 + p_2)^2 = E_{\text{com}}^2 .$$

<u>claim:</u> $\mathcal{M}^\dagger = \frac{e^2}{s} (\bar{v}_4 \gamma^\mu u_3)(\bar{u}_1 \gamma^\mu v_2)$

<u>Pf:</u> $\gamma_\mu^\dagger \gamma_0 = \gamma_0 \gamma_\mu .$

$$\Rightarrow (\bar{\Psi}_1 \gamma^\mu \Psi_2)^\dagger = \bar{\Psi}_2 \gamma^\mu \Psi_1$$

$\blacksquare$

$$\Rightarrow |\mathcal{M}_{\mu^+\mu^- \leftarrow e^+e^-}|^2 = \frac{e^4}{s^2} \left((\bar{u}_3 \gamma^\nu v_4)(\bar{v}_4 \gamma^\mu u_3) \right)_{\mu\mu} \times \left((\bar{u}_1 \gamma_\mu v_2)(\bar{v}_2 \gamma_\nu u_1) \right)_e$$

<u>UNPOLARIZED scattering</u>: (Average over initial & Sum over final)

$$\sum_{s_4} v_a^{s_4}(p_4) \bar{v}_b^{s_4}(p_4) = (\not{p}_4 - m_\mu)_{ab}$$

$$\sum_{S_3} u_d^{s_3}(p_3)\,\bar{u}_a^{s_3}(p_3) = (\not{p}_3 + m_\mu)_{da}$$

$$\to \sum_{S_3 S_4} \left(\bar{u}(p_3)_a^{s_3}\,\gamma^\mu_{ab}\,V(p_4)_b^{s_4}\right)\left(\underbrace{\bar{V}(p_4)_c^{s_4}}_{(\not{p}_4 - m_\mu)_{bc}}\,\gamma^\nu_{cd}\,u^{s_3}(p_3)_d\right)$$

$$= \gamma^\mu_{ab}\,(\not{p}_4 - m_\mu)_{bc}\,\gamma^\nu_{cd}\,(\not{p}_3 + m_\mu)_{da}$$

$$= \mathrm{tr}\left(\gamma^\mu (\not{p}_4 - m_\mu)\gamma^\nu (\not{p}_3 + m_\mu)\right)$$

$$= p_{4\rho}\,p_{3\sigma}\,\underbrace{\mathrm{tr}\,\gamma^\mu\gamma^\rho\gamma^\nu\gamma^\sigma} - m_\mu^2\,\underbrace{\mathrm{tr}\,\gamma^\mu\gamma^\nu}$$

$$\pm 2 m_\mu\,\mathrm{tr}\,\gamma^\mu\gamma^\rho\gamma^\nu\,p_\rho \quad 0$$

SPINOR TRACE NINJUTSU :

- $\mathrm{tr}(AB\cdots C) = \mathrm{tr}(CAB\cdots)$ — cyclicity of trace

- $\mathrm{tr}\,\mathbb{1}_{4\times 4} = 4.$

$(\gamma^5)^2 = \mathbb{1}.$

$\{\gamma^5, \gamma^\mu\} = 0.$

- $\underline{\mathrm{tr}\,\gamma^\mu} = \mathrm{tr}(\gamma^5)^2\gamma^\mu$

$$\overset{\text{cot}}{=} \mathrm{tr}\,\gamma^5 \underbrace{\gamma^\mu \gamma^5}_{= -\gamma^5\gamma^\mu}$$

$$= -\mathrm{tr}\,(\gamma^5)^2\gamma^\mu = -\underline{\mathrm{tr}\,\gamma^\mu}.$$

- $\mathrm{tr}(\gamma)^{\text{odd}} = 0.$

- $\mathrm{tr}\,\gamma^\mu\gamma^\nu \overset{\text{clifford}}{=} \mathrm{tr}(2\eta^{\mu\nu} - \gamma^\nu\gamma^\mu)$

$$\overset{\text{cot}}{=} 2\eta^{\mu\nu}\cdot 4 - \mathrm{tr}\,\gamma^\mu\gamma^\nu$$

$$\Rightarrow \boxed{\mathrm{tr}\,\gamma^\mu\gamma^\nu = 4\eta^{\mu\nu}.}$$

- $\mathrm{tr}\,\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = 4\left(\eta^{\mu\nu}\eta^{\rho\sigma} + \eta^{\sigma\mu}\eta^{\rho\nu} - \eta^{\mu\rho}\eta^{\nu\sigma}\right)$

<u>Pf:</u> $\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma \propto \gamma^5$

$\mathrm{tr}\,\gamma^5 = 0.$

<u>if $\mu = \nu$ then $\rho = \sigma$.</u> and $\longrightarrow 4$

$(\gamma^\mu)^2 = \pm \mathbb{1}$

$\mathrm{tr}\,\gamma^\mu\gamma^\mu\gamma^\rho\gamma^\sigma \propto \mathrm{tr}\,\gamma^\rho\gamma^\sigma = 4\eta^{\rho\sigma}.$

- $\mathrm{tr}\,\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5 = -4i\epsilon^{\mu\nu\rho\sigma}.$ $\lrcorner$

$$\sum_{\text{spins}} (\bar{u}\, \gamma^\nu v)(\bar{v}\, \gamma^\mu u)_{\text{muon}}$$

$$= 4\left(p_4^\mu p_3^\nu + p_3^\nu p_4^\mu - p_{34}\, \eta^{\mu\nu} - m_\mu^2\, \eta^{\mu\nu} \right).$$

$$p_{34} \equiv p_3 \cdot p_4 . \equiv p_3^\mu (p_4)_\mu .$$

<u>claim</u>: unknown data in initial state must be averaged.

$$\rho_{\text{initial}} = \sum_{s_1, s_2} \rho_{s_1, s_2}\, p_{s_1, s_2}$$

<u>$\mathrm{tr}(\rho_{\text{initial}}) = 1$.</u> $\longrightarrow$ $\sum_{s_1, s_2} p_{s_1, s_2} = 1.$

$\Rightarrow$ <u>$p_{s_1 s_2} = \frac{1}{4}$.</u>

$$\rho_{s_1 s_2} = | s_1\, s_2 \rangle\langle s_1, s_2 |$$

$$\frac{1}{4}\sum_{s_1..s_4}|M|^2 = \frac{e^4}{4s^2}\left(\mathrm{tr}\,\gamma^\mu(\not p_4 - m_\mu)\gamma^\nu(\not p_3 + m_\mu)\right)\times$$

$$\left(\mathrm{tr}\,\gamma_\nu(\not p_2 - m_e)\gamma_\mu(\not p_1 + m_e)\right)$$

$$\overset{\text{algebra}}{=} \frac{8e^4}{s^2}\left(p_{13}p_{24} + p_{14}p_{23} + m_\mu^2 p_{12} + m_e^2 p_{34} + 2m_e^2 m_\mu^2\right).$$

$$= \frac{2e^4}{s^2}\left(t^2 + u^2 + 4s(m_e^2 + m_\mu^2) - 2(m_e^2 + m_\mu^2)^2\right)$$

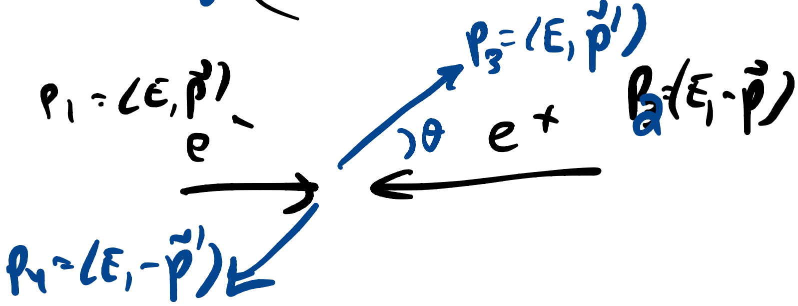
$$\left(\frac{d\sigma}{d\Omega}\right)_{com} = \frac{1}{64\pi^2 E_{com}^2}\frac{|\vec p'|}{|\vec p|}\left(\frac{1}{4}\sum_{spin}|M|^2\right)$$

($s+t+u = \sum_i m_i^2$

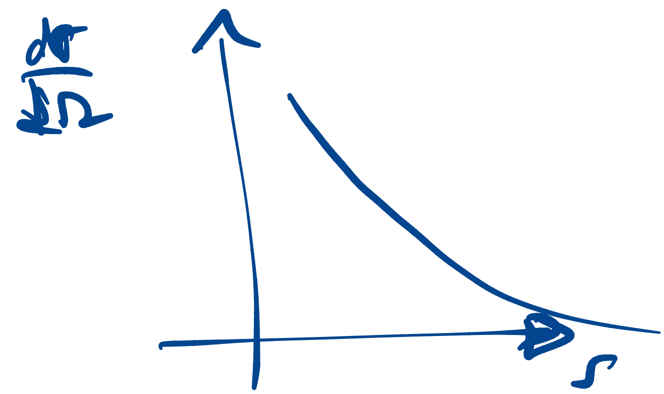
2 indep kinematic vars.)

$$= \frac{\alpha^2}{64\pi^2}\frac{|\vec p'|}{|\vec p|}\left(E^4 + |\vec p'|^2|\vec p'|^2\cos^2\theta + E^2(m_e^2 + m_\mu^2)\right)$$

$$\left(\begin{aligned} E^2 &= \vec p^2 + m_e^2 \\ E^2 &= \vec p'^2 + m_\mu^2. \end{aligned}\right).$$



$$\xrightarrow[\gg m_\mu^2]{E \gg m_e^2} \quad \frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_{cm}^2}\left(1+\cos^2\theta\right).$$



$$\gamma_W^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$$

$$\tilde{\gamma}_W = \begin{pmatrix} 0 & \tilde{\sigma} \\ -\tilde{\sigma} & 0 \end{pmatrix} = i\sigma^y \otimes \tilde{\sigma}$$

$$\gamma_W^0 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix} = \sigma^x \otimes \mathbb{1}.$$

$$i\sigma^y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma^x \sigma^y = i\sigma^z \quad \ldots$$

$$\sigma^x \sigma^y (\sigma^x)^\dagger = -\sigma^y.$$

$$\tilde{\gamma}^0 = \sigma^z \otimes \mathbb{1}.$$

$$\mathcal{U} \gamma^0 \mathcal{U}^\dagger \stackrel{!}{=} \tilde{\gamma}^0$$

$$\mathcal{U} (\sigma^x \otimes \mathbb{1}) \mathcal{U}^\dagger = \sigma^z \otimes \mathbb{1}.$$

$$\mathcal{U} (i\sigma^y \otimes \tilde{\sigma}) \mathcal{U}^\dagger = i\sigma^y \otimes \tilde{\sigma}$$

hint: $\mathcal{U} = A \otimes B.$ $\quad \begin{cases} A\sigma^x A^\dagger = \sigma^z \\ B\tilde{\sigma} B^\dagger = \tilde{\sigma} \end{cases}$

$$\partial\left(\underbrace{\psi_L^{\alpha} \sigma^{\mu}_{\alpha\dot\alpha} \chi_R^{\dot\alpha}}_{(\frac{1}{2},0)\otimes(0,\frac{1}{2}) = (\frac{1}{2},\frac{1}{2})} \right) = \partial\psi\, \sigma\chi + \psi\sigma\partial\chi$$

$$= \ldots$$

$\psi_L \uparrow (\frac{1}{2},0)$ $\qquad$ $\chi_R \uparrow (0,\frac{1}{2})$

Dirac $\quad (\frac{1}{2},0) \oplus (0,\frac{1}{2})$

$\qquad\qquad \psi_L \qquad\qquad \psi_R$

$$\mathcal{L}_{\text{Dirac}} = \underbrace{\psi_L^{\dagger} i\sigma^{\mu}\partial_{\mu}\psi_L}_{\left((\frac{1}{2},\frac{1}{2})\otimes(\frac{1}{2},\frac{1}{2})\right)\ldots = (0,0).} + \underbrace{\psi_R^{\dagger} i\bar\sigma\partial_{\mu}\psi_R}$$

$\psi_L \in (\frac{1}{2},0)$

$$\underline{\psi_L^{\dagger} i\sigma^{\mu} \partial_{\nu}\psi_L}$$