5.5 Quantum Spinor Fields

Solutions to free Dirac eqn: $(i\partial\!\!\!/ - m)\Psi(x) = 0$

$$\Psi(x) = e^{-ip^\mu x_\mu}\, u(\vec{p}) \quad \Big| \; p^0 = \omega_p > 0.$$

w/ $(p\!\!\!/ - m)\, u(p) = 0 \implies$

$$u(\vec{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi \\ \sqrt{p\cdot\bar{\sigma}}\,\xi \end{pmatrix}$$

FT in space $\longrightarrow$ 4 ODEs.
Linear first-order

$\rightarrow$ 4 lin. indep solns for each $\vec{p}$.

like KG:

1 linear 2d-order ODE

$\rightarrow$ 2 soln's for each $\vec{p}$

e^{-ipx} w/ $p^0 = \pm\omega_p = \pm\sqrt{\vec{p}^2 + m^2}$

Negative-energy sol^{n}s: $\Psi = e^{+ipx} v(\vec{p})$

w/ $p^0 > 0$

& $p^2 = m$

$\longrightarrow (-\not{p} - m) v(\vec{p}) = 0.$

$$\Longrightarrow \quad v^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\eta \\ -\sqrt{p\cdot\bar{\sigma}}\,\eta \end{pmatrix}$$

ξ, η are 2-component spinors.

Normalization & Completeness : $\bar{u} \equiv u^\dagger \gamma^0$

$$\begin{cases} \bar{u}_r(\vec{p})\, u_s(\vec{p}) = 2m\, \xi_r^\dagger \xi_s \stackrel{!}{=} 2m\,\delta_{rs} \\ \bar{v}_r(\vec{p})\, v_s(\vec{p}) = -2m\, \eta_r^\dagger \eta_s \stackrel{!}{=} -2m\,\delta_{rs}. \end{cases}$$

(vs: $u_r^\dagger(\vec{p})\, u_s(\vec{p}) = 2\omega_{\vec{p}}\, \xi_r^\dagger \xi_s$ not Lorentz Invt)

For each $\vec{p}$: $\partial_t \Psi = \underbrace{\gamma^0(\vec{\gamma}\cdot\vec{p} + m)}_{\equiv\, h_D}\Psi$ 1-particle Dirac hamiltonian

$h_D(\vec{p}) = h_D(\vec{p})^\dagger$

for each $\vec{p}$: • $\bar{u}^r(\vec{p})v^s(\vec{p}) = \bar{v}^r(\vec{p})u(\vec{p}) = 0.$

(**But**: $u^{r\dagger}(\vec{p})v^s(\vec{p}) \neq 0.$)

But but: $\Psi = e^{+ipx}v(\vec{p})$

$$= e^{-iqx}v(-\vec{q})$$

$$p = -q.$$

$\Longrightarrow$ • $u_r^{\dagger}(\vec{p})\,V_s(-\vec{p}) = 0.$

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choose: $\displaystyle\sum_{s=1,2} \xi_s \xi_s^{\dagger} \overset{\star}{=} \mathbb{1}_2 = \sum_{s=1,2} \eta_s \eta_s^{\dagger}$

$$\Longrightarrow \sum_{s=1,2} u^s(\vec{p})\,\bar{u}^s(\vec{p}) = \sum_s \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi \\ \sqrt{p\cdot\bar{\sigma}}\,\xi \end{pmatrix} \left(\xi_s^{\dagger}\sqrt{p\cdot\sigma},\ \xi_s^{\dagger}\sqrt{p\cdot\bar{\sigma}}\right)$$

$$\overset{\star}{=} \begin{pmatrix} \sqrt{p\cdot\sigma}\sqrt{p\cdot\bar{\sigma}} & (\sqrt{p\cdot\sigma})^2 \\ (\sqrt{p\cdot\bar{\sigma}})^2 & \sqrt{p\cdot\bar{\sigma}}\sqrt{p\cdot\sigma} \end{pmatrix} = \begin{pmatrix} m & p\cdot\sigma \\ p\cdot\bar{\sigma} & m \end{pmatrix}$$

$$\underbrace{(p\cdot\sigma)(p\cdot\bar{\sigma})}_{=p^2\mathbb{1}}$$

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$$

$$\Rightarrow \boxed{\begin{aligned} \sum_{s=1,2} u^s(\vec{p})\,\bar{u}^s(\vec{p}) &= \gamma^\mu p_\mu + m \\ &= \not{p} + m \end{aligned}}$$

(like $\displaystyle\sum_{s=1,2} e^s_i(\vec{p})\, e^{s*}_j(\vec{p}) = \delta_{ij} - \hat{p}_i \hat{p}_j$)

(<u>also</u>: $\displaystyle\sum_{s=1,2} v^s(\vec{p})\,\bar{v}^s(\vec{p}) = \not{p} - m.$)

<u>Q</u>: What did we need to build Fock space?

for each mode, $N = a^\dagger a$ w/

$$\left\{ \begin{array}{ll} [N, a] = -a & , \; [N, a^\dagger] = a^\dagger \\ \text{"a lowers"} & \text{"}a^\dagger\text{ raises"} \end{array} \right] \; ✱$$

Given $N|n\rangle = n|n\rangle$

$$\Rightarrow \begin{cases} N(a|n\rangle) = (n-1)(a|n\rangle) \\ N(a^\dagger|n\rangle) = n+1\,(a^\dagger|n\rangle) \end{cases}$$

$n \geq 0$ since $0 \leq \| a^\dagger |n\rangle \|^2 = \langle n|N|n\rangle$

unitary $\quad \forall n \quad = n\langle n|n\rangle.$

$\Rightarrow \exists n_0$ s.t. $a|n_0\rangle = 0$

$\Rightarrow n_0|n_0\rangle = N|n_0\rangle$

$= a^\dagger \underbrace{a|n_0\rangle}_{=0} \Rightarrow n_0 = 0$

$\Rightarrow$ spectrum of $N \in \{0, 1, 2 \ldots\}$.

I didn't use $[a, a^\dagger] = 1$.

If $\boxed{a^\dagger a + a a^\dagger \equiv \{a, a^\dagger\} = 1 \qquad \{a,a\} = 0 = \{a^\dagger, a^\dagger\}}$

$$\begin{cases} |0\rangle \text{ w } a|0\rangle = 0 \\ |1\rangle = a^\dagger|0\rangle \text{ w } a^\dagger|1\rangle = 0. \end{cases} \Longleftarrow \quad (\text{ie } a^2 = 0)$$

[Pauli]

claim: $\{a, a^\dagger\} = 1 \quad a^2 = 0, (a^\dagger)^2 = 0$

$$\Longrightarrow \quad [N, a] = -a \quad , \quad [N, a^\dagger] = a^\dagger .$$

Pf: $[AB, C] = A\{B, C\} - \{A, C\}B$.

Multiple Modes:
$$\begin{cases} \{a_i, a_j^\dagger\} = \delta_{ij} \\ \{a_i, a_j\} = 0 \end{cases}$$

$$a_i^\dagger a_j^\dagger |0\rangle = +|00\ldots0\,1_i\,0\ldots0\,1_j\ldots0\rangle$$

$$= -a_j^\dagger a_i^\dagger |0\rangle \qquad [\text{Fermi}] .$$

Anticommuting scalar fields: $\phi(x) = \int \frac{d^dp}{\sqrt{2\omega_p}} \left(a_p e^{-ipx} + h.c.\right)$

$$w/ \quad \pi = \dot{\phi}$$

$$\longrightarrow H = \frac{1}{2}\int \left(\pi^2 + (\nabla\phi)^2 + m^2\phi^2\right) = \int dp \, \frac{\omega_p}{2} \left(\underbrace{a_p^\dagger a_p + a_p a_p^\dagger}_{= \delta_{p,p} = \mathbb{1}}\right)$$

what if $\{a_p, a_q^\dagger\} = \delta_{p,q}$

Cx scalar: $\Phi = \int \frac{d^d p}{\sqrt{2\omega_p}} \left(a_p e^{-ipx} + b_p^\dagger e^{ipx} \right)$

$$\to H \underset{\substack{\uparrow \\ \text{no assumptions}}}{=} \int d^d p \; \frac{1}{2} \omega_p \left(a_p^\dagger a_p + b_p b_p^\dagger \right)$$

$$\overset{\{b, b^\dagger\} = \delta}{=} \int dp \; \frac{\omega_p}{2} \left(a_p^\dagger a_p \underbrace{- b_p^\dagger b_p}_{} + \text{cst}\,\mathbb{1} \right)$$

unbounded below $\times$.

Dirac Hamiltonian

$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\Psi}} = i \bar{\Psi} \gamma^0$
$= i \Psi^\dagger$

$$\Rightarrow h = \pi \dot{\Psi} - \mathcal{L}$$

$$= \bar{\Psi} \left(i \vec{\gamma} \cdot \vec{\nabla} + m \right) \Psi$$

$$\overset{\text{eom}}{=} \Psi^\dagger i \partial_t \Psi$$

$$\begin{cases} \Psi(x) = \int \frac{d^3 p}{\sqrt{2\omega_p}} \sum_{s=1,2} \left(a_{\vec{p}}^s u^s(\vec{p}) e^{-ipx} + b_{\vec{p}}^{s\dagger} v^s(\vec{p}) e^{+ipx} \right) \\ \bar{\Psi}(x) = \int \frac{dp}{\sqrt{2\omega}} \sum_s \left(a^{s\dagger} \bar{u}(\vec{p}) e^{ipx} + b \bar{v} e^{-ipx} \right) \end{cases}$$

$$\rightarrow H = \int d^3x \, h$$

$$= \int_x \int_p \int_q \sum_{s,s'} \left(u_p^{s\dagger} e^{ipx} a_p^\dagger + v^{s\dagger} e^{-ipx} b_p \right) \times$$

$$\left(\omega_q \, u^s(q) e^{-iqx} a_q^s - \omega_q v^s(q) e^{iqx} b_q^\dagger \right)$$

$$= \int_p \int_q \underbrace{\left(u^\dagger u - v^\dagger v \right)}_{\int_x \to \delta(p-q)} + \underbrace{\left(v^\dagger u - u^\dagger v \right)}_{\delta(p=-q)}$$

$$u_s^\dagger(p) u_{s'}(p) = v_s^\dagger(p) v_{s'}(p)$$

$$= 2\omega_p \delta_{ss'}$$

$$v_r^\dagger(p) u_s(-p) = 0.$$

$$\rightarrow H = \int d^3p \, \omega_p \sum_s \left(a_p^{s\dagger} a_p^s - b_p^s b_p^{s\dagger} \right)$$

IF $\{ b_s(p), b_{s'}(q)^\dagger \} = \delta^d(\vec{p}-\vec{q})$

$$\rightarrow H = \int d^3p \, \omega_p \sum_s \left(N_s^a(p) + N_s^b(p) \right) + \text{cst.}$$

$$\longrightarrow H - E_0 \geq 0.$$

Comments on spin-statistics:

In rel QFT $\frac{1}{2}$-integer spin fields $\longrightarrow$ fermions

integer " " $\longrightarrow$ bosons

$$\mathcal{L}_{\chi+\frac{1}{2}} \ni \bar{\Psi}_{\alpha\rho\ldots}\, \gamma^{\mu}_{\alpha\rho}\, \partial_\mu \Psi_{\rho}{}^{\rho\ldots}$$

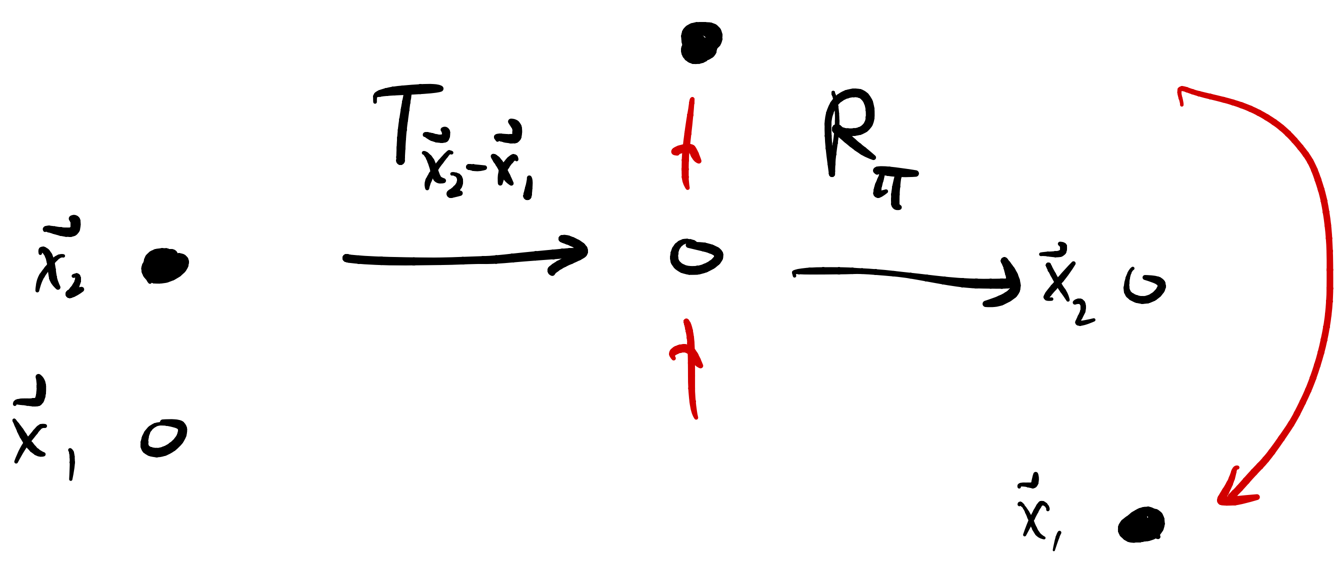
$$\longrightarrow h \ni \bar{\Psi}\, \dot{\Psi}$$

$$\mathcal{L}_{\chi} \ni \phi_{\rho\ldots}\, \partial_\mu \partial^\mu \phi^{\rho\ldots} \longrightarrow h \ni \dot{\phi}\dot{\phi}$$

$$\widehat{SWAP}\, |\phi_1(x_1)\, \phi_2(x_2)\rangle = \underline{\underline{e^{i\phi_K}}}\, |\phi_1(x_2)\, \phi_2(x_1)\rangle$$

$\uparrow \qquad \uparrow$

<u>How</u>:



on a Dirac particle at rest: $\Lambda(\theta) = \exp\left(-i\,\theta\cdot J^{\mu\nu}_{\mathrm{Dirac}}\right)$

$$\Lambda(\vec{\theta} = \theta_z \check{z}, \beta = 0) = \exp\left(i\theta_z\, \sigma^3 \otimes \mathbb{1}_{L,R}\right)$$

$$= \begin{pmatrix} e^{i\theta^z/2} & & & \\ & e^{-i\theta^z/2} & & \\ & & e^{i\theta^z/2} & \\ & & & e^{-i\theta^z/2} \end{pmatrix} \overset{\theta^z = \pi}{=} \begin{pmatrix} i & & & \\ & -i & & \\ & & i & \\ & & & -i \end{pmatrix}$$

$$= i(-1)^{\text{spin}}$$

Suppose $\phi_1(x_1), \phi_2(x_2)$ are Dirac particles of the same definite spin S^z.

$$\phi_1 = (1\ 0\ 1\ 0) \longmapsto \Lambda\phi_1 = (i\ 0\ i\ 0)$$
$$= i\phi_1$$

$$\Lambda(\vec{\theta} = \pi\hat{z}) T_{x_2 - x_1} |\phi_1(x_1)\,\phi_2(x_2)\rangle = (\pm i)^2 |\phi_1(x_2)\phi_2(x_1)\rangle$$

$$= -|\phi_1(x_2)\phi_2(x_1)\rangle$$

$\Longrightarrow$ fermions.

Pert. Thy &

Dirac Propagator:

as for scalars but:
① signs
② matrices.

$$T\left(A_1(x_1)\cdots A_n(x_n)\right) \equiv \underline{\underline{(-1)^p A_{1'}(x_{1'})\cdots A_{n'}(x_{n'})}}$$

w/ $x_{1'}^0 > x_{2'}^0 > \dots x_{n'}^0$

w/ $\boxed{p \equiv \#\text{ of fermion interchanges required to go from } 1\dots n \text{ to } 1'\dots n'.}$

$$:ABC\cdots: \equiv (-1)^p \underbrace{A'B'C'}_{\text{ann. ops to the right}}$$

$$:a_p a_q a_r^+: = (-1)^2 a_r^+ a_p a_q$$

$\underline{\{a_p, a_q\} = 0}$.

$$= (-1)^3 a_r^+ a_q a_p$$

<u>WICK</u>: $T(ABC\cdots) = :ABC\cdots: + \sum(\text{contractions}).$

$$\overbrace{\Psi(x)\,\Psi(y)} = 0 \qquad \overbrace{\bar\Psi(x)\,\bar\Psi(y)} = 0$$

$$\overbrace{\Psi(x)\,\bar\Psi(y)} = S_F(x-y)$$

$$S_F^{ab}(x-y) \equiv \langle 0|\, T\left(\Psi^a(x)\,\bar\Psi^b(y)\right)|0\rangle$$

$a,b = 1..4$

$$= \theta(x^0-y^0)\langle 0|\Psi^a(x)\bar\Psi^b(y)|0\rangle$$

$$-\theta(y^0-x^0)\langle 0|\bar\Psi^b(y)\Psi^a(x)|0\rangle$$

$$\boxed{\Psi \sim a + b^\dagger = \Psi^{(+)} + \Psi^{(-)}}$$

$$= \theta(x^0-y^0)\underbrace{\langle 0|\Psi_a^{(+)}(x)\bar\Psi_b^{(-)}(y)|0\rangle}_{=\langle 0|\{\Psi^+,\,\bar\Psi^-\}|0\rangle \equiv S^+}$$

only a's $\longrightarrow$

$$-\theta(y^0-x^0)\underbrace{\langle 0|\bar\Psi_b^{(-)}(y)\Psi_a^{(+)}(x)|0\rangle}_{=\langle 0|\{\bar\Psi^-,\,\Psi^+\}|0\rangle \equiv S^-}$$

only b's $\longrightarrow$

$$S^{(+)}_{ab}(x-y) = \{\Psi^{+}_{a}(x), \bar{\Psi}^{-}_{b}(y)\}$$

$$= \int \frac{d^3p}{\sqrt{2\omega_p}} e^{-ipx} \sum_{s=1,2} \int \frac{d^3q}{\sqrt{2\omega_q}} e^{-iqx} \sum_{s'=1,2}$$

$$u^s_a(p)\, \bar{u}^{s'}_b(q) \underbrace{\{a^s_p, a^{s'\dagger}_q\}}_{=\,\delta^d(p-q)\,\delta^{ss'}}$$

$$(p^0 = \omega_p)$$

$$= \int \frac{d^3p}{2\omega_p} e^{-ip(x-y)} \underbrace{\sum_s u^s_a(p)\, \bar{u}^s_b(p)}_{(\not{p}+m)_{ab}}$$

$$= \int \frac{d^3p}{2\omega_p} \left(i\not{\partial}_x + m\right)_{ab} e^{-ip(x-y)}$$

$$= (i\not{\partial}_x + m)_{ab} \underbrace{\int \frac{d^3p}{2\omega_p} e^{-ip(x-y)}}_{\Delta^+(x-y) \,=\, \int_{C_+} d^4p\, e^{-ip(x-y)} \frac{i}{p^2-m^2}}$$

$$S^+ = \int_{C^+} d^4p \, e^{-ip(x-y)} \frac{i(\not{p}+m)_{ab}}{p^2-m^2}$$

$$S^-_{ab}(x-y) = \int_{C^-} \left(\ldots \right)$$

$$\Rightarrow S_F^{ab}(x-y) = \int_{C_F} \left(\ldots \right)$$

$$\Rightarrow \tilde{S}_F(p) = \frac{i(\not{p}+m)}{p^2-m^2+i\epsilon} = \frac{i(\not{p}+m)}{(\not{p}+m)(\not{p}-m)+i\epsilon}$$

$$= \frac{i}{\not{p}-m+i\epsilon}.$$

__Note:__ $\Psi(x)$ does __not__ commute w/ $\bar{\Psi}(y)$ when $(x-y)^2 < 0$ (spacelike)!

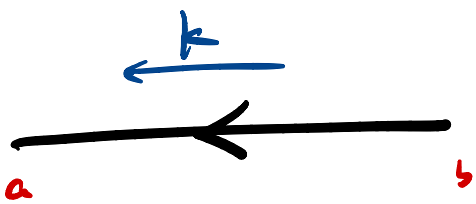
Rather, $\underline{\{\Psi(x), \bar{\Psi}(y)\} = 0}$.

we can measure $\bar{\Psi}\gamma^{\cdots}\Psi = \mathcal{O}_B$

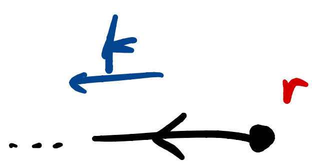
and $[\mathcal{O}_B^{(x)}, \mathcal{O}_B^{(y)}] = 0$.

we can't measure Ψ.

Feynman Rules w/ Spinors $\left(\xleftarrow{t}\right)$

① 

$= \left(\dfrac{i}{\not{k} - m_\Psi}\right)_{ab}$

② 

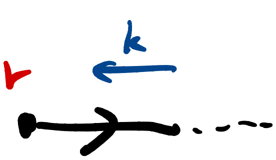
$= \Psi|k,r\rangle = u^r(k)$ particle in initial state

③ 

$= \langle k,r|\bar{\Psi} = \bar{u}^r(k)$ " " final "

④ 

$= \bar{\Psi}|k,r\rangle = \bar{v}^r(k)$ antiparticle in initial state

⑤ 

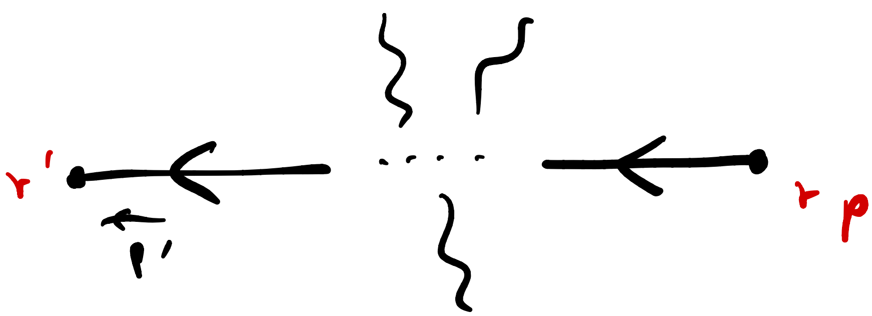
$= \langle k,r|\Psi = v^r(k)$ " " final state

(6) advice: start at the end of a fermion line + keep going ...

Why: fermion parity: fermion lines cannot end.

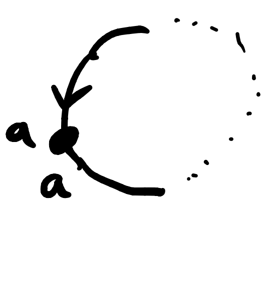
($\Psi \to -\Psi$ is an invariance.)

$\mathcal{L}$ is an even power of Ψ.



$$= \sum_{a,b=1..4} \bar{u}^{r'}(p')_a \left(\begin{matrix}\text{pile of}\\ \gamma \text{ matrices}\end{matrix}\right)_{ab} u^r(p)_b$$

(note: $(\not{p}-m)u=0$.)



$$\propto \sum_a \left(\text{pile of } \gamma \text{ matrices}\right)_{aa} = \mathrm{tr}(\text{pile}).$$

7. Diagrams related by swapping ext. fermion lines have a relative (-1).

8. a loop of fermions $\propto (-1)$.

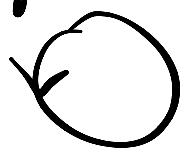
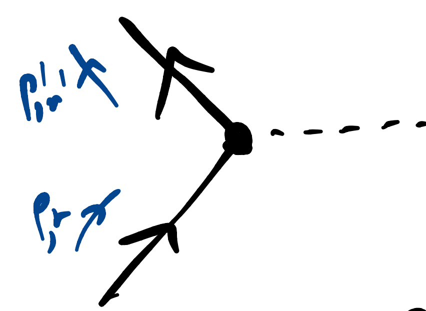


ILLUSTRATION: $\mathcal{L}_{Yukawa} = \mathcal{L}_{Dirac} + \mathcal{L}_{KG} + \mathcal{L}_{int}$

$$\mathcal{L}_{int} = - g \, \phi \, \bar{\Psi} \, \Psi$$

pion meson

nucleon



$= -ig\delta^{rr'}$

Note: if $\langle 0 | \phi | 0 \rangle = v$

then $m_\Psi = m_0 + g v$.

$$\left[\begin{aligned} \gamma^\mu \gamma^\nu &= \frac{1}{2}\{\gamma^\mu, \gamma^\nu\} + \frac{1}{2}[\gamma^\mu, \gamma^\nu] \\ &= \eta^{\mu\nu} \mathbb{1} + \frac{1}{2}\underbrace{[\gamma^\mu, \gamma^\nu]}_{\propto J^{\mu\nu}} \end{aligned}\right.$$

$$[J^{\mu\nu}, \gamma^\rho] = \frac{i}{4}\left[[\gamma^\mu, \gamma^\nu], \gamma^\rho\right]$$

$$= \frac{i}{4}\Big(\mu\nu\rho - \rho\mu\nu$$

$$- \nu\mu\rho + \rho\nu\mu \Big)$$