4.2 S-matrix (cont'd)

eg: $\underline{\phi \sim a + a^\dagger}$ $\qquad V = g\phi\Phi\Phi^*$

$\Phi \sim b + \underline{c^\dagger}$

$\Phi^* \sim c + \underline{b^\dagger}$

$|i\rangle \sim a^\dagger_p |0\rangle \qquad |f\rangle \sim b^\dagger_{q_1} c^\dagger_{q_2} |0\rangle$

"meson" "nucleon - antinucleon pair"

$$S_{fi} = \langle f| T e^{-i\int_{-\infty}^{\infty} d^D x\, V(\phi_I(x))} |i\rangle$$

$$= -ig \langle f| T \int d^D x\, \phi_x \Phi_x \Phi^*_x |i\rangle + O(g^2)$$

$$= -ig (2\pi)^D \delta^{D+1}(\underline{q_1 + q_2 - p})$$

only nonzero when

In the rest frame of ϕ: $p^\mu = (M, \vec{0}) \Rightarrow \vec{q}_1 = -\vec{q}_2$

and $\underline{M = 2\sqrt{|\vec{q}_1|^2 + m^2} \geq 2m.}$

$$P_{fi} \sim |S_{fi}|^2 = g^2 \left(\delta^D(p_f - p_i)\right)^2 = \infty.$$

$$= g^2\, \delta^D(p_f - p_i)\, \delta^D(0)$$

$$= g^2\, \delta^D(p_f - p_i) \underbrace{\int d^D x\, e^{i x \cdot 0}}_{V \cdot T}$$

finite $\dfrac{\text{PROB}}{\text{unit time}\cdot\text{unit vol}}$ = rate.

$\longrightarrow$ lifetime of ϕ.

4.3 Wick's Thm

$$\hat{S} = T\, \underbrace{e^{-i\int_{-\infty}^{\infty} dt \int d^d x\, V(\phi_I(x,t))}}$$

wick: $T(\phi \cdots \phi) = \underbrace{:\phi \cdots \phi:}_{\text{"normal ordered product"}} + \underline{\underline{\text{full contractions}}}$

$$\phi(x) = \phi^{+}(x) + \phi^{-}(x)$$
$$\sim \quad a \quad + a^{+}$$

$$:\phi(x)\,\phi(y): \;\equiv\; \phi^{-}(x)\,\phi^{+}(y) + \phi^{-}(y)\,\phi^{+}(x) + \phi^{-}(x)\,\phi^{-}(y) + \phi^{+}(x)\,\phi^{+}(y)$$

Q: what is $:c\mathbb{1}: \equiv c$.

⤳ this def $:\;:$ is not a linear operator.

$$:a^{+}a: \;=\; a^{+}a$$

$$= :(a a^{+} - 1): \;\overset{?}{=}\; :a a^{+}: - \underline{:\mathbb{1}:}$$

$$= a^{+}a \;-\; :\mathbb{1}:$$

$$\Longrightarrow \underline{:\mathbb{1}: \overset{?}{=} 0.}$$

if it were linear $\langle 0| : \text{anything} : |0\rangle \overset{?}{=} 0.$

INSTEAD:

$$:ABC\cdots: \;=\; \big((\text{only } a^{+}\text{s})\;(\text{only } a\text{ s})\big)$$

$$\langle 0| : \left(\begin{matrix}\text{anything except}^{+} \\ a\ (-\mathbb{1}\end{matrix}\right) : |0\rangle = 0.$$

⌈ fermions : $c_k c_p^{\dagger} + c_p^{\dagger} c_k \equiv \{ c_k, c_p^{\dagger} \}$

$= \delta(k-p)$

$\{ c_k^{\dagger}, c_p^{\dagger} \} = 0$

($(c_p^{\dagger})^2 = 0$ is Pauli principle !)

$: ABC \ldots :$

$= \left(\left(\text{only } a^{\dagger}\text{'s} \right) \left(\text{only } a\text{'s} \right) \right) (-1)^P$

$P \equiv$ # of fermion interchanges along the way.

$$:\phi_x \phi_y: = T(\phi_x \phi_y) + \theta(x^0 - y^0) \underbrace{[\phi^-(y), \phi^+(x)]}_{= -\Delta^+(x-y)}$$

$$+ \theta(y^0 - x^0) \underbrace{[\phi^-(x), \phi^+(y)]}_{= -\Delta^+(y-x)}$$

c-# !

$$\boxed{:\phi_x \phi_y: = T(\phi_x \phi_y) - \Delta_F(x-y).}$$

$$\equiv T(\phi_x \phi_y) - \underbrace{\phi_x \phi_y}$$

"contraction"

$$:\phi_1 \cdots \phi_n: \;=\; T\,\phi_1 \cdots \phi_n \;-\; (\text{all contractions})$$

eg: $$T(\phi_1 \ldots \phi_4) = :\phi_1 \cdots \phi_4 + \Big(\overbrace{\phi_1\phi_2}\phi_3\phi_4 + \overset{5}{\text{more}}\Big):$$

$$+\Big(\overbrace{\phi_1\phi_2}\,\overbrace{\phi_3\phi_4} + \underbrace{\overbrace{\phi_1\,\overbrace{\phi_2\,\phi_3}\,\phi_4} + \overbrace{\phi_1\,\phi_2\,\phi_3}\!\overbrace{\phi_4}}\Big)$$

$$\langle 0|\,T\,\phi_1 \ldots \phi_4\,|0\rangle = \Delta_F(12)\,\Delta_F(34) + 2 \text{ more.}$$

"full contractions".

Pf: $x_1^0 \geq x_2^0 \geq \cdots \geq x_m^0$

$$T(\phi_1 \ldots \phi_m)$$

$$= \phi_1\, T(\phi_2 \ldots \phi_m) \overset{\text{wick for } 2..m}{=}$$

$$\phi_1\Big(:\phi_2 \cdots \phi_m: + \begin{matrix}\text{all contractions}\\ \text{w/o } \phi_1\end{matrix}\Big)$$

$\phi_1^- + \phi_1^+$ $\nearrow$

$$= :\phi_1^- \phi_2 \cdots \phi_m: + \begin{matrix}\text{all contractions}\\ \text{w } \phi_1\end{matrix}.$$

4.4 time-ordered correlators & diagrams

$$G^{(n)}(x_1 \ldots x_n) = \langle \Omega | T \, \phi_1^H(x_1) \cdots \phi_n^H(x_n) \, | \underset{\uparrow}{\Omega} \rangle$$

$$\tilde{G}^{(n)}(p_1 \ldots p_2) = \int d^Dx_1 \ldots \int d^Dx_n \, e^{-i\sum p_i x_i} \, G^{(n)}(x_1 \ldots x_n)$$

$$\boxed{\begin{aligned} H|\Omega\rangle &= E_0|\Omega\rangle \\ H &= H_0 + V \end{aligned}}$$

In the free thy $V = 0$:

$$G^{(2)}_{free}(x_1 x_2) = \Delta_F(x_1 - x_2) \equiv$$

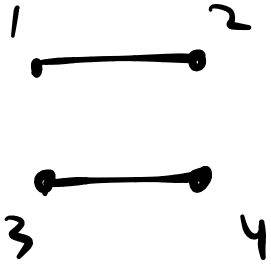
$$\tilde{G}^{(2)}_{free}(p_1, p_2) = \delta^D(p_1 + p_2) \frac{i}{p_1^2 - m^2 + i\epsilon}$$

$$G^{(4)}_{free}(x_1 \ldots x_4) = \langle 0 | T \, \phi_1 \cdots \phi_4 | 0 \rangle$$

$$\overset{\text{wick}}{=} \cancelto{0}{\langle 0 | : \phi_1 \ldots \phi_4 : | 0 \rangle} + \sum(\text{contractions})$$

$$G^{(4)}_{free} = \Delta_F(12)\,\Delta_F(34) + \Delta_F(13)\,\Delta_F(24) + \Delta_F(14)\,\Delta_F(23)$$

=



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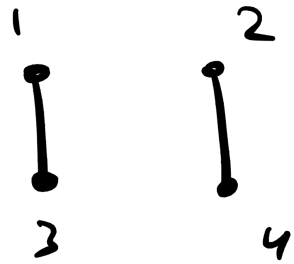

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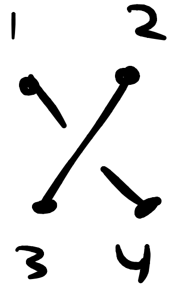
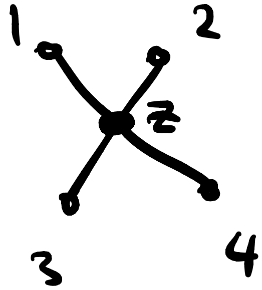


<u>eg</u>: $V = \int d^4z \, \frac{\lambda}{4!}\phi^4(z)$

<u>expect</u>: $G^{(4)} = G^{(4)}_{free} +$



$+ \mathcal{O}(\lambda^2)$

<u>preview</u>: $$\tilde{G}^{(n)}(p) \overset{p_i^2 \to m_i^2}{\sim} \prod_i \frac{i}{p_i^2 - m_i^2 + i\epsilon} \underset{\underset{\text{S-matrix element}}{\uparrow}}{S(p_1 \cdots p_n)}$$

[LSZ]

$$0 = \int D\phi \, \frac{\delta}{\delta\phi_x}\left(\phi_1 \cdots \phi_{n-1} \, e^{iS[\phi]} \right)$$

$$\overset{\text{fue}}{=} (\Box + m^2) \, G^{(n)}(x \, x_1 \ldots x_{n-1})$$

$$- \delta(x-x_1) \, G^{(n-2)}(x_2 \ldots x_{n-1})$$

$$- \delta(x-x_2) \, G^{(n-2)}(x_1 x_3 \ldots x_{n-1})$$

$$- \ldots$$

$\Longrightarrow$ Wick's thm. [see Schwartz].

Perturbative expansion of $G^{(n)}$. strategy:

① relate $|\Omega\rangle$ to $|0\rangle$

② Relate ϕ_H to ϕ_I

③ Wick.

- Fix $H_0|0\rangle = 0$.

- $$\mathbb{1} = \sum_n |n\rangle\langle n| = |\Omega\rangle\langle\Omega| + \sum_{n \neq \Omega} |n\rangle\langle n|$$

  eigenstates of $H = H_0 + V$.

- assume $\langle\Omega|0\rangle \neq 0$.

Step 1 : $\langle 0| e^{-iHT}$

$$= \langle 0| \sum_n \underline{|n\rangle\langle n|}\, \underline{e^{-iHT}}$$

$$= \underline{\underline{\langle 0|\Omega\rangle\langle \Omega| e^{-iE_0T}}} + \underline{\underline{\sum_{n\neq\Omega} \langle 0|n\rangle\langle n| e^{-iE_nT}}}$$

now let $T \to \infty(1-i\epsilon)$ (euclidean time evolution)

$$E_0 < E_n$$

$$\Rightarrow \left| e^{-iE_0T} \right| >> \left| e^{-iE_nT} \right| \quad \forall n \neq \Omega.$$

$$\langle \Omega| = \lim_{T\to\infty(1-i\epsilon)} \left(\frac{\langle 0| e^{-iHT} e^{iE_0T}}{\langle 0|\Omega\rangle} \right)$$

$\underline{\langle 0|H_0 = 0}$

$$= \lim_{T\to\infty(1-i\epsilon)} \left(\frac{\langle 0| \overset{= U_I(T)}{\boxed{e^{iH_0T} e^{-iHT}}} e^{iE_0T}}{\langle 0|\Omega\rangle} \right)$$

replace $T \to \underline{T - t_0}$

$$\lim_{T\to\infty(1-i\epsilon)} \left(\frac{\langle 0| \underline{U_I(T, t_0)}\, e^{+iE_0(T-t_0)}}{\langle 0|\Omega\rangle} \right)$$

$$|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{U_I(t_0,-T)|0\rangle}{e^{-iE_0(T+t_0)}\langle\Omega|0\rangle}.$$

step 2:

$$\phi_I = U_0^\dagger \phi U_0$$

$$\phi_H = U_H^\dagger \phi U_H$$

$$\Rightarrow \phi_H = U_H^\dagger U_0 \phi_I U_0^\dagger U_H$$

$$= U_I^\dagger \phi_I U_I.$$

Assemble: $G^{(2)}(x,y) = \langle\Omega| T\, \phi_H(x)\phi_H(y)|\Omega\rangle$

$\boxed{\text{If } x^0 > y^0}$

$$= \lim_{T\to\infty(1-i\epsilon)} \left(e^{-iE_0(T-t_0)}\langle 0|\Omega\rangle\right)^{-1} \langle 0| U_I(T,t_0)$$

$$U_I^\dagger(x^0,t_0)\phi_I(x) \underbrace{U_I(x^0,t_0)\cdot U_I^\dagger(y^0,t_0)}_{=U_I(x^0,y^0)}\phi_I(y) U_I(y^0,t_0)$$

$$\underbrace{U_I(t_0,-T)|0\rangle}_{=U_I(y^0,-T)} \left(e^{-iE_0(T+t_0)}\langle\Omega|0\rangle\right)^{-1}.$$

$$= \lim_{T\to\infty(1-i\epsilon)} \left(e^{-2iE_0T}|\langle 0|\Omega\rangle|^2\right)^{-1}$$

$$\underbrace{\langle 0| U_I(T,x^0)\,\phi(x)\,U_I(x^0,y^0)\,\phi(y)\,U_I(y^0,-T)|0\rangle}$$

already time ordered!

$$= \langle 0|\,T\,\phi(x)\phi(y)\,U_I(T,-T)|0\rangle$$

using $U_I(T,x^0)U_I(x^0,y^0) = U_I(T,y^0)$

- also true if $y^0 > x^0$.
- $G^n(x_1\ldots x_n) = \lim \dfrac{\langle 0|\,T\,\phi_1\ldots\phi_n\,U\,|0\rangle}{(\quad D\quad)}$

what's D?

$D =$

$$1 = \langle\Omega|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \underline{\left(e^{-2iE_0T}|\langle 0|\Omega\rangle|^2\right)^{-1}} \times$$

$$\underbrace{\langle 0|\;U(T,t_0)\,U(t_0,-T)|0\rangle}$$

$$= U(T,-T)$$

$$= Te^{-i\int_{-T}^{T} V}$$

$$\Rightarrow \quad D(T) = \langle 0 | T\, e^{-i\int_{-T}^{T} V}$$

$$\Rightarrow \quad \boxed{G^{(n)}(x_1 \ldots x_n) = \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0 | T\, \phi_1 \ldots \phi_n\, e^{-i\int_{-T}^{T} V(t')dt'} |0\rangle}{\langle 0 | T\, e^{-i\int_{-T}^{T} V(t')dt'} |0\rangle}}$$

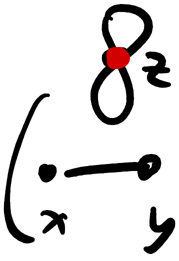
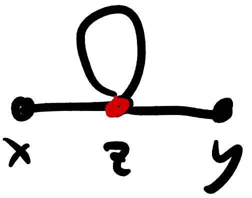
examples: $V = \frac{\lambda}{4!}\phi^4$

$$G^{(2)}_{num}(x,y) \equiv \langle 0 | T\, \phi_x \phi_y\, e^{-i\int d^D z \frac{\phi^4(z)\lambda}{4!}} |0\rangle$$

$$= \langle 0 | T\, \phi_x \phi_y |0\rangle - \frac{i\lambda}{4!}\int d^D z\, \langle 0 | T\left(\phi_x \phi_y\, \phi_z \phi_z \phi_z \phi_z\right)|0\rangle$$

wick

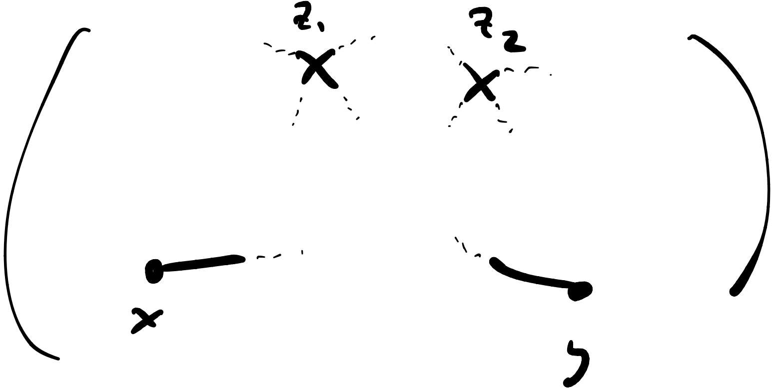
$$= \Delta_F(x-y) - \frac{i\lambda}{4!}\int d^D z \left(\underset{\times 3}{\wick{\c1\phi_x \c1\phi_y \c2\phi_z \c2\phi_z \c3\phi_z \c3\phi_z}} + \underset{\times 4 \cdot 3}{\wick{\c1\phi_x \c2\phi_y \c1\phi_z \c2\phi_z \c3\phi_z \c3\phi_z}} \right) + \mathcal{O}(\lambda^2)$$

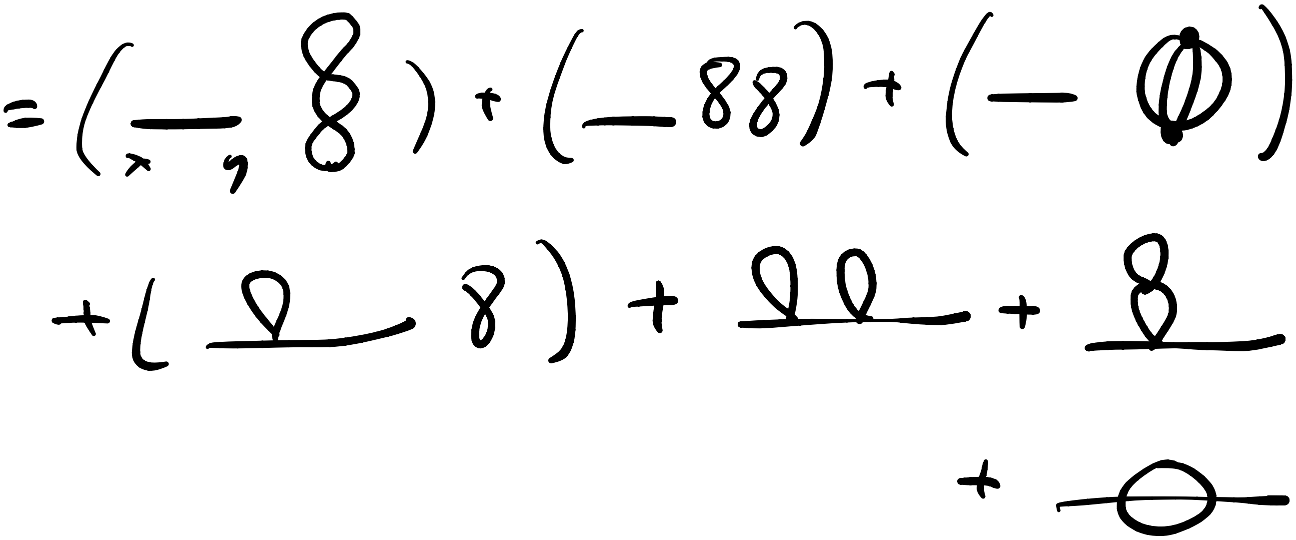
$=$  $+$ () $+$ () $+\mathcal{O}(\lambda^2)$

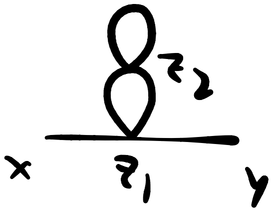
↑ fully connected

$O(\lambda^2)$ bit: $\quad \frac{1}{2!}\left(\frac{-i\lambda}{4!}\right)^2 \int d^D z_1 \int d^D z_2 \langle 0| T\, \phi_x \phi_y\, \phi_{z_1}^4\, \phi_{z_2}^4 |0\rangle$

$= \sum$







$$\propto \quad \overbrace{\phi_x \phi_{z_1}}\ \overbrace{\phi_y \phi_{z_1}}\left(\overbrace{\phi_{z_1}\phi_{z_2}}\right)^2\left(\overbrace{\phi_{z_2}\phi_{z_2}}\right)$$

Feynman rules for ϕ^4 thy in pos'n space

$$\{\text{diagrams}\} \equiv \{A\} = \{A_0\} \cup \{A_1\} \cup \{A_2\} \ldots$$

$\{A_m\}$ = all diagrams m internal 4-pt vertices

one external vertex for each x_i

$$G^{(n)}(x_1 \ldots x_n) = \sum_A \underset{\sim}{\mathcal{M}_A}$$

<u>To find $\mathcal{M}_A$:</u>

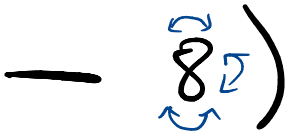
- put a $-i\lambda \int d^D z_a$ for internal vertex.

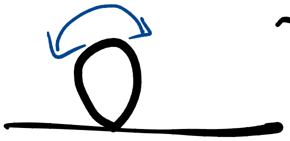
- put a $\Delta_F(y_i - y_j)$ for each edge

$y \in \{x_i, z_a\}$

- multiply by $s(A) = |\mathrm{Aut}(A)|^{-1}$.

$|\underline{\mathrm{Aut}(A)}|$ = # of ways of permuting the ingredients of A that map A to itself.

$\underline{\varphi}$: $s($  $) = \frac{1}{4!}3 = \frac{1}{8}$

$s($  $) = \frac{1}{4!}4\cdot 3 = \frac{1}{2}$

$$\int dx \int dy \quad \underbrace{T\,\phi_x^4\,\phi_y^4}_{}$$

$$\theta(x^0-y^0)\,\phi_x^4\,\phi_y^4$$

$$\underline{\underline{+\,\theta(y^0-x^0)\,\phi_y^4\,\phi_x^4}}$$

Relabel $\tilde{x}^0 = y^0$

$\tilde{y}^0 = x^0$

$$= \int dx \int dy \;\; \underline{\underline{\theta(x^0-y^0)}}\,\phi_x^4\,\phi_y^4$$

$$+ \int d\tilde{x} \int d\tilde{y} \;\; \theta(\tilde{x}^0-\tilde{y}^0)\,\phi_{\tilde{x}}^4\,\phi_{\tilde{y}}^4$$