4.1 Time evolution, continued.

${}_p\langle \varphi, t | \mathcal{O}_p(t) | \psi, t\rangle_p$ is indep. of P. *

If $H = \underset{\uparrow \text{ solvable}}{H_0} + \underset{\text{"small"}}{V}$:

$\boxed{P = I}$ "Interaction picture" is useful

$$\left[\mathcal{O}_I(t) \equiv e^{+iH_0 t}\, \mathcal{O}_I(0)\, e^{-iH_0 t} \right]$$

$${}_I\langle \varphi, t | \mathcal{O}_I(t) | \psi, t\rangle_I \overset{!}{=} {}_H\langle \varphi, t | \mathcal{O}_H(t) | \psi, t\rangle_H$$

$$= \langle \varphi, 0 | U^\dagger(t)\, \mathcal{O}_H\, U(t) | \psi, 0\rangle$$

PRICE:

- $|\psi, t\rangle_I = U_I(t) |\psi, 0\rangle$ w/ $\underline{U_I(t) = U_0^\dagger(t)\, U(t)}$

$$\left(\begin{aligned} U(t) &\equiv U_H(t) \\ &\equiv e^{-iHt} \end{aligned} \right)$$

- $i \frac{d}{dt} V_I(t) = [V_I, H_0]$ or $V_I(t) = U_0^\dagger(t) V_I U(t)$

$\underline{\text{eg}}$: $V_I = \int d^dx \; g \; \underbrace{\phi(x)}_{=\phi(x,0)}{}^3$

$\equiv V_I(0)$

$\underline{\underline{1 = U_0 U_0^\dagger}}$

$\underline{U_0(t) \equiv e^{-itH_0}}$

$$V_I(t) = U_0^\dagger V_I U_0$$

$$= g \int d^dx \; U_0^\dagger \phi^3(x,0) U_0$$

$$= g \int d^dx \; \underbrace{U_0^\dagger \phi(x,0) U_0}_{\phi_I(x,t)} \underbrace{U_0^\dagger \phi(x,0) U}_{\phi_I(x,t)} \underbrace{U_0^\dagger \cdots}_{\phi_I(x,t)}$$

$$= g \int d^dx \;\; \phi_I(x,t)^3 \, .$$

$\underline{\text{Conclusion}}$: $\left(V(t)\right)_I = V\Big|_{t=0}\left(\phi_I(t)\right)$.

$$U_I(t) = U_0^\dagger(t) U(t) = e^{iH_0t} e^{-iHt}$$

$$\overset{?!}{\neq} e^{i(H_0-H)t}$$

$[H_0,H]=0$

$$= e^{-iVt}$$

$$\partial_t U_I(t) = \qquad (\sim \partial_t |\psi, t\rangle_I = \dots)$$

$$|\psi, t\rangle_I = \underline{U_I(t)} |\psi, 0\rangle$$

$$\partial_t \left(U_0^\dagger(t) \underbrace{U_H(t)}_{e^{-iH(0)t} = e^{-i(H_0+V)t}} \right)$$

$$= U_0^\dagger(t) \underbrace{(iH_0 - iH)}_{= -iV} U_H(t)$$

$$\partial_t e^{iAt} = iA \cdot e^{iAt} = e^{iAt} \cdot iA$$

Note: we assume

$$H = H_0 + V$$

is time-independent

$$\partial_t H = 0.$$

$$\Rightarrow [H, H] = 0. \quad \forall t.$$

$$|\psi, t\rangle_S = e^{-iHt} |\psi, 0\rangle$$

$$= -i\, U_0^\dagger(t)\, V(0) \wedge U_H(t)$$

$$\mathbb{1} = U_0(t) U_0^\dagger(t)$$

$$= -i \underbrace{U_0^\dagger(t) V(0) U_0(t)}_{= V_I(t)} \cdot \underbrace{U_0^\dagger(t) U_H(t)}_{U_I(t)}$$

$$= -i\, V_I(t)\, U_I(t)$$

$$\phi_I(x, t) = \int \frac{d^dp}{\sqrt{2\omega_p}} \left(e^{-ipx} a_p + \text{h.c.} \right)$$

Conclusion: $i\partial_t U_I(t) = V(t)\, U_I(t)$

$$\boxed{\text{like } \quad \partial_t f(t) = \alpha(t) f(t) \quad \Rightarrow f(t) = e^{\int^t \alpha(t')dt'}}$$

But: $[V(t), V(t')] \neq 0$.

Peskin's notation: $U_I(t) \equiv U(t, t_0)\Big|_{t_0=0}$

$$\left.\begin{aligned} |\psi, t\rangle_I &= U_I(t)\,|\psi, 0\rangle \\ |\psi, t'\rangle_I &= U_I(t')\,|\psi, 0\rangle \end{aligned}\right\} \Rightarrow |\psi, 0\rangle = U_I^\dagger(t')\,|\psi, t'\rangle$$

$$\Rightarrow |\psi, t\rangle_I = \underbrace{U_I(t)\, U_I^\dagger(t')}_{\equiv U(t,t')}|\psi, t'\rangle_I$$

$$\Rightarrow U(t, t') = e^{iH_0 t}\, e^{-iH(t-t')}\, e^{-iH_0 t'}.$$

From now on: everything is I.

3-step plan for particle physics:

① At time t_i prepare a state of {particles}

$$|i\rangle \equiv |\Psi, t_i\rangle \in \mathcal{H}_{QFT}$$

② Wait. At time t the state is

$$U(t, t_i)\,|i\rangle = |\Psi, t\rangle.$$

③ At time t_f measure all the particle labels $\longrightarrow |f\rangle$

(momenta, spins, types ...)

QM says

$$\mathrm{Prob}(f \leftarrow i) = \left|\langle f|\, U(t_f, t_i)\,|i\rangle\right|^2$$

$$\xrightarrow[t_f \to +\infty]{t_i \to -\infty} \left|\langle f|\, \underbrace{U(+\infty, -\infty)}_{\equiv \hat{S}}\,|i\rangle\right|^2$$

$$= |S_{fi}|^2 \equiv |S\text{-matrix}|^2$$

Dyson expansion:

(linear 1st order ODE)

Let's solve $\partial_t |\psi,t\rangle = -i V(t)|\psi,t\rangle$

w/ initial condition $|\psi,t_i\rangle = |i\rangle$.

Solution: *

$$|\psi,t\rangle = |i\rangle + (-i)\int_{t_i}^{t} dt_1\, V(t_1)\, |\psi,t_1\rangle$$

state at time $t \geq t_i$

state at time $t_1 \in [t_i, t]$.

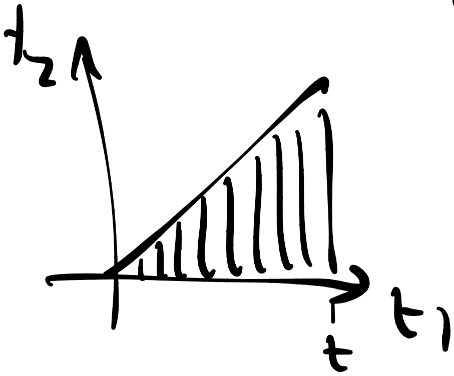
Plug * in here

$$|\psi,t\rangle = |i\rangle + (-i)\int_{t_i}^{t} dt_1\, V(t_1)\Big[\overbrace{|i\rangle + (-i)\int_{t_i}^{t_1} dt_2\, V(t_2)\,|\psi,t_2\rangle}^{*(t_1)}\Big]$$

$$= |i\rangle + (-i)\int_{t_i}^{t} dt_1 V(t_1)|i\rangle$$

$$+ (-i)^2 \int_{t_i}^{t} dt_1 \int_{t_i}^{t_1} dt_2\, \underbrace{V(t_1)V(t_2)}_{\text{is time ordered.}}\, \underbrace{|\psi,t_2\rangle}_{\text{don't know}}$$

Plug * in here



$t_2 \leq t_1$

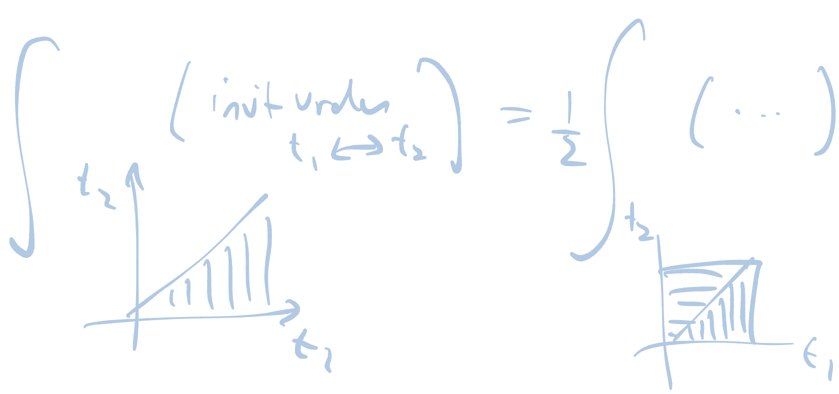
$$|\psi, t\rangle = \sum_{n=0}^{\infty} (-i)^n \int_{t_i}^{t} dt_1 \int_{t_i}^{t_1} dt_2 \int_{t_i}^{t_2} \cdots \int_{t_i}^{t_{n-1}} dt_n \, V(t_1)\, V(t_2) \cdots V(t_n)\, |i\rangle$$

$$U(t, t_i) = U(t, t_i)\,|i\rangle. \quad \underline{\forall i}.$$

$$= \mathcal{T}\left(\sum_{n=0}^{\infty} (-i)^n \int_{t_i}^{t} dt_1 \int_{t_i}^{t_1} dt_2 \int_{t_i}^{t_2} \cdots \int_{t_i}^{t_{n-1}} dt_n \, V(t_1)\, V(t_2) \cdots V(t_n) \right)$$

$$= \mathcal{T}\left(\sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{t_i}^{t} dt_1 \int_{t_i}^{t} dt_2 \int_{t_i}^{t} \cdots \int_{t_i}^{t} dt_n \, V(t_1)\, V(t_2) \cdots V(t_n) \right)$$

$$= \mathcal{T} \, e^{-i \int_{t_i}^{t} V(t')\,dt'} .$$

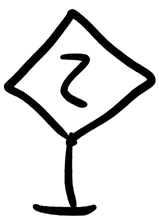


4.2 S-matrix

$$\hat{S} = U(-\infty,\infty) = T\left(e^{-i\int_{-\infty}^{\infty} dt\, V(t)}\right)$$

$$= \sum_n \frac{(-i)^n}{n!}\, T\left(\int_{-\infty}^{\infty} dt\, V(t)\right)^n$$

Known fn of $\phi_I(t) = \sum_p (a_p + a_p^\dagger)$



$|i\rangle, |f\rangle$?

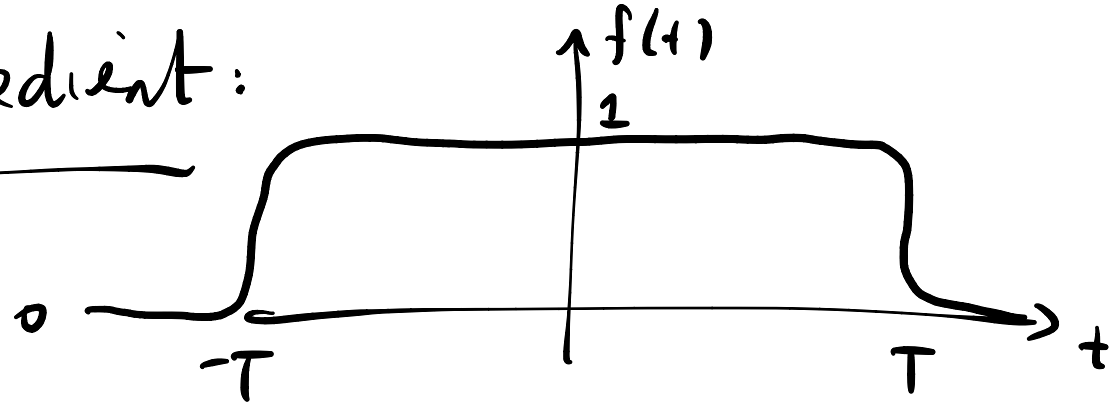
$|i\rangle \overset{\text{eg}}{=} |$ a state w/ a particle of momentum $p\rangle = \sqrt{2\omega_p}\, a_p^\dagger |0\rangle$

$a_p^\dagger|0\rangle$, $|0\rangle$ is eigenstate of H_0

$a_p^\dagger|0\rangle$, $|0\rangle$ is not an eigenstate of H.

Let $|\Omega\rangle$ = groundstate of H.

expedient:



Replace V with $f(t)V(t)$.

$t_i < -T \qquad t_f > T$.

(WRONG but useful.)

example: "scalar Yukawa theory"

$$\mathcal{L} = \underbrace{\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{2}M^2\phi^2}_{} + \tfrac{1}{2}\partial_\mu\Phi\partial^\mu\Phi^* - \tfrac{1}{2}m|\Phi|^2$$

$$+ \mathcal{L}_I$$

ϕ mediates a force of range $1/M$

$$\mathcal{L}_I \equiv -g\,\phi\,\Phi\,\Phi^*.$$

$g\,\Phi\Phi^* \sim$ Force of giant hand.

$$\phi(x) = \int \frac{d^d p}{\sqrt{2\omega_p}} \left(a_p e^{-ipx} + a_p^{+} e^{ipx} \right) \Big|_{p^0=\omega_p}$$

$$\omega_p \equiv \sqrt{\vec{p}^2 + M^2}$$

$$\Phi(x) = \int \frac{d^d p}{\sqrt{2E_p}} \left(b_p e^{-ipx} + c_p^{+} e^{ipx} \right) \Big|_{p^0=E_p}$$

$$E_p \equiv \sqrt{\vec{p}^2 + m^2}$$

<u>Note</u>: $\Phi \to e^{i\alpha}\Phi$ is a $U(1)$ symmetry.

$$q = N_c - N_b$$

This is crude caricature of nuclear physics.

$$b^{+}|0\rangle \sim |\text{proton}\rangle$$

"scalar"

$$a^{+}|0\rangle \sim |\text{pion}\rangle$$

Artisanal Meson decay:

$$|i\rangle = \sqrt{2\omega_p}\, a_p^\dagger |0\rangle$$

$$|f\rangle = \sqrt{2E_{q_1} 2E_{q_2}}\, b_{q_1}^\dagger c_{q_2}^\dagger |0\rangle$$

S_{fi} describes the decay amplitude

$$\phi \to \Phi\, \bar{\Phi}$$

$a|0\rangle = b|0\rangle = c|0\rangle = 0$

$$S_{fi} = \langle f|\hat{S}|i\rangle =$$

all at time x^0

$$\langle f| T\Big(1 - (-i)\int d^{d+1}x\, g\, \underbrace{\phi(x)\Phi^\dagger(x)\Phi(x)}_{x} + \mathcal{O}(g^2)\Big)|i\rangle$$

$\phi(x)$

leading nonzero term:

$$-ig\int d^{d+1}x\, \langle f|\Phi^\dagger_x \Phi_x \left(\int \frac{d^dk}{\sqrt{2\omega_k}} e^{-ikx} a_k + \cancel{h.c.}\right) \sqrt{2\omega_p}\, a_p^\dagger|0\rangle$$

$\langle 0| b_{q_1} c_{q_2}$

$a_k a_p^\dagger|0\rangle = (2\pi)^d \delta^d(k-p)|0\rangle$

$$= -ig \int d^{d+1}x \, \langle 0 | b_{q_1} c_{q_2} \sqrt{YEE}$$

$$\left(\int \frac{d^d k_1}{\sqrt{2E_{k_1}}} e^{ik_1 x} b^\dagger_{k_1} \right) \left(\int \frac{d^d k_2}{\sqrt{2E_{k_2}}} e^{ik_2 x} c^\dagger_{k_2} \right)$$

$$e^{-ipx} \, | 0 \rangle$$

$$= -ig \int d^{d+1}x \; e^{i(q_1 + q_2 - p)x}$$

$$= -ig \, (2\pi)^{d+1} \, \delta^{d+1} \underline{(q_1 + q_2 - p)}$$

$\neq 0$ if

$\underline{M \geq 2m.}$

$$\begin{cases} p^0 = \omega_p = \sqrt{\vec{p}^2 + M^2} \\ q^0_{1,2} = E_{\vec{q}_{1,2}} = \sqrt{\vec{q}^2_{1,2} + m^2} \end{cases}$$

$$0 = \int dq \, \frac{\partial}{\partial q} \left(\text{Anything} \cdot e^{-S(q)} \right)$$

$$= \langle 1 \rangle + c_1 \langle q^2 \rangle + c_2 \langle q^4 \rangle$$

$$\langle q^2 \rangle \propto -\frac{\partial}{\partial m^2} Z(m^2)$$

$$\text{eqn} \equiv Z''(x) + f(x) Z'(x) + g(x) Z(x) == 0.$$

$$Z(x) \equiv e^{+x^2/a} (x^2)^{+1/4} K(x^2/a)$$

DSolve[eqn , Z , x]

$$0 = \int D\phi \frac{\delta}{\delta \phi(x)} \left(\phi(y) e^{iS[\phi]} \right) \Rightarrow (\Box + m^2) \langle q | \phi(x) \phi(y) | q \rangle = \delta^4(x-y)$$

$$S(\varphi) = \cdots \, g \, \varphi^4$$

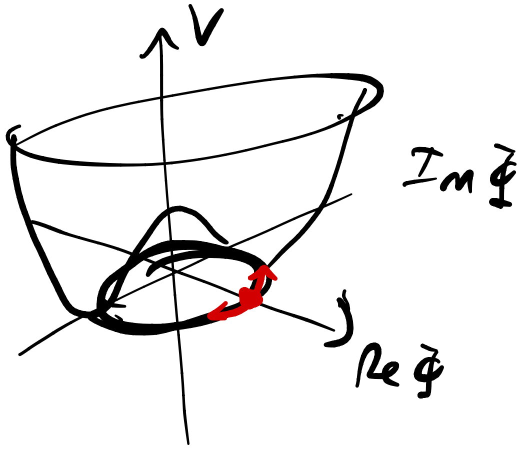
$$\text{vs} \quad \frac{\tilde{g}}{4!} \varphi^4$$

$$Z(g) = Z(\tilde{g}/4!)$$

$$= \sum_n c_n g^n = \sum_n c_n \left(\frac{\tilde{g}}{4!}\right)^n$$

$$G = \sum \left(\begin{array}{c} \text{connected diagrams} \\ \text{w/ 2 legs} \end{array} \right)$$

$$Z = \sum \left(\begin{array}{c} \text{all diagrams} \\ \text{w/ 0 legs} \end{array} \right)$$



$$V = \lambda(|\Phi|^2 - v^2)^2$$

$$- \delta(\Phi + \Phi^*)$$

$$\Phi = \rho e^{i\phi}$$

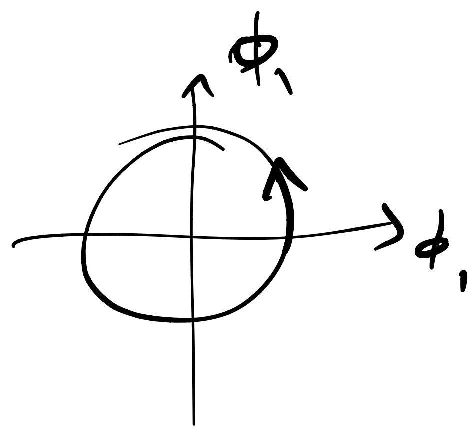
$$V(\phi) = \delta\rho \cos\phi$$

$$m_\phi^2 \sim \delta\rho$$

$$\delta \Phi_\alpha(x) = -\frac{1}{2} i \epsilon^i \sigma^i_{\alpha\beta} \Phi_\beta(x)$$

$$\underline{\epsilon^i = \epsilon \hat{n}^i}$$

$$\begin{cases} \delta\phi_1 = \epsilon i \phi_2 \\ \delta\phi_2 = -i\epsilon\phi_1 \end{cases}$$



$$T_{\mu\nu} = g_{\mu\sigma} g_{\nu\rho} T^{\sigma\rho}$$

$$ds^2 = -dt^2 + a^2(t)\, d\vec{x}^2$$

$$m^2 \to m^2 a(t)^2$$

$$\langle 0 | T_{ii} | 0 \rangle \overset{?}{\propto} \langle 0 | (\partial_i \phi)^2 | 0 \rangle$$

$$= \int d^d k \, \frac{k_i^2}{\omega_k}$$

$$= \frac{1}{d} \int d^d k \, \frac{k^2}{\omega_k}$$

$$= \frac{1}{d} \int d^d k \left(\frac{\omega_k^2}{\omega_k} - \frac{m^2}{\omega_k} \right)$$

$$= \frac{1}{d} \int d^d k \, \omega_k - \frac{m^2}{d} \int \frac{d^d k}{\omega_k}$$