3. Lorentz Invariance & causality

Relativistic normalization of 1-particle state:

$$\text{Fock space} = \text{span}\left\{ |\vec{p}_1, \vec{p}_2 \dots\rangle \propto a^\dagger_{p_1} a^\dagger_{p_2} \dots |0\rangle \right\}$$

$a_p|0\rangle = 0$

$$\langle 0|0\rangle \overset{!}{=} 1 .$$

$$|\vec{p}\rangle = c_p \, a^\dagger_p |0\rangle$$

What if $c_p \overset{?}{=} 1$?

$$\langle \vec{k}|\vec{p}\rangle \overset{?}{=} \langle 0| a_k a^\dagger_p |0\rangle \overset{a_k|0\rangle = 0}{=}$$

$$\langle 0| [a_k, a^\dagger_p] |0\rangle$$

in frame F.

$$= (2\pi)^d \delta^d(k-p) \equiv \cancel{\delta}(k-p)$$

$$1 = \underline{\int d^d p} \; \underline{\cancel{\delta}(k-p)} \quad \text{is Lorentz invariant}$$

$$p'_\mu = \Lambda_\mu{}^\nu p_\nu$$

Boost in x direction:

$$\begin{pmatrix} E' \\ p_x' \\ p_y' \\ p_z' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & & \\ -\beta\gamma & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} E \\ p_x \\ p_y \\ p_z \end{pmatrix}$$

$$\Rightarrow \quad \frac{dp_x'}{dp_x} = \gamma\Big(1 - \beta \underbrace{\frac{dE}{dp_x}}_{= p_x/E}\Big) = \gamma\left(1 - \beta\frac{p_x}{E}\right)$$

Lorentz sym: $E^2 = \vec{p}^{\,2} + m^2 = p_x^2 + p_\perp^2 + m^2 \Rightarrow 2E\frac{dE}{dp_x} = 2p_x$

$$\frac{dp_x'}{dp_x} = \frac{\gamma}{E}(E - \beta p_x) = \frac{E'}{E}.$$

$$\delta^{(d)}(\vec{p} - \vec{k}) = \frac{d^d p'}{d^d p}\,\delta^{(d)}(p' - k') \overset{\substack{\text{boost} \\ \text{in } x}}{=} \frac{dp_x'}{dp_x}\,\delta^{d}(p' - k')$$

$$= \frac{E'}{E}\,\delta^{(d)}(\vec{p}\,' - \vec{k}')$$

$$\Rightarrow \langle \vec{k}' | p' \rangle \overset{?!}{=} \frac{E'}{E}\,\delta(p' - k')$$

Instead: $|\vec{p}\rangle = \sqrt{2\omega_p}\, a_{\vec{p}}^\dagger |0\rangle.$

$\omega_p = \sqrt{\vec{p}^2 + m^2}.$

$$\Rightarrow \langle \vec{k} | \vec{p} \rangle = \sqrt{4\omega_k \omega_p}\, \delta(k-p)$$

$$= 2\omega_p\, \delta(k-p)$$

$$\langle \vec{k}' | \vec{p}' \rangle = \cancel{2\omega_p} \cdot \frac{\omega_{p'}}{\cancel{\omega_p}}\, \delta(k'-p')$$

i.e. $|\vec{p}'\rangle = \sqrt{2\omega_{p'}}\, a_{p'}^\dagger |0\rangle$ ✓.

why $\sqrt{2}$? Identity in the 1-particle $\mathcal{H}$ is

$$\mathbb{1}_1 = \int \frac{d^d k}{2\omega_{\vec{k}}} |\vec{k}\rangle\langle\vec{k}|$$

$$= \int d^d k \int dk_0\, \theta(k_0) \frac{\delta\left(k^0 - \sqrt{\vec{k}^2 + m^2}\right)}{2k^0} |\vec{k}\rangle\langle\vec{k}|$$

$$= \int d^{d+1}k\, \theta(k^0)\, 2\pi\, \delta(k^2 - m^2)\, |\vec{k}\rangle\langle\vec{k}|$$

$$\boxed{\delta(f(x)) = \sum_{x_0 | f(x_0) = 0} \frac{\delta(x - x_0)}{|f'(x_0)|}}$$

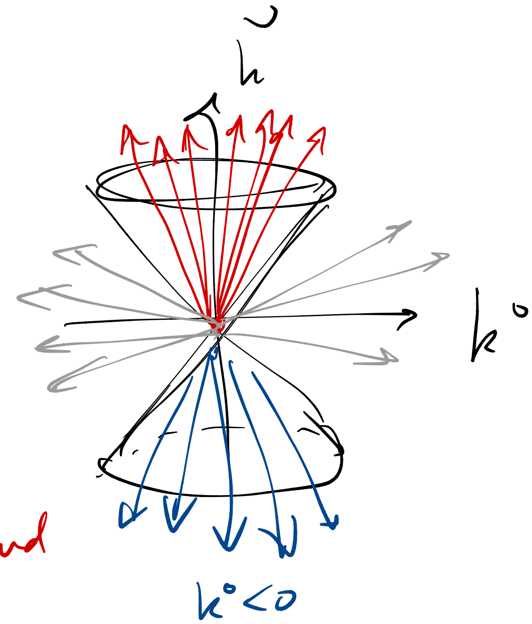
$k_0^2 - \vec{k}^2 - m^2$

$k_0 = \pm \omega_{\vec{k}}$

MANIFESTLY Lorentz inv't:

- $k^\mu k_\mu$
- $|\vec{k}\rangle$
- $\underline{\theta(k^0)} \rightarrow \theta(k^0)$

orbit of k^μ under Lorentz if $k^\mu k_\mu > 0$ and $k^0 > 0$ stays in forward lightcone



($\mathrm{sign}(k^0)$ of a timelike 4-vector is Lorentz inv't.)

Is it Lorentz inv't? $\quad [\phi(\vec{x}), \pi(\vec{y})] = i\,\delta^{(d)}(\vec{x}-\vec{y})$

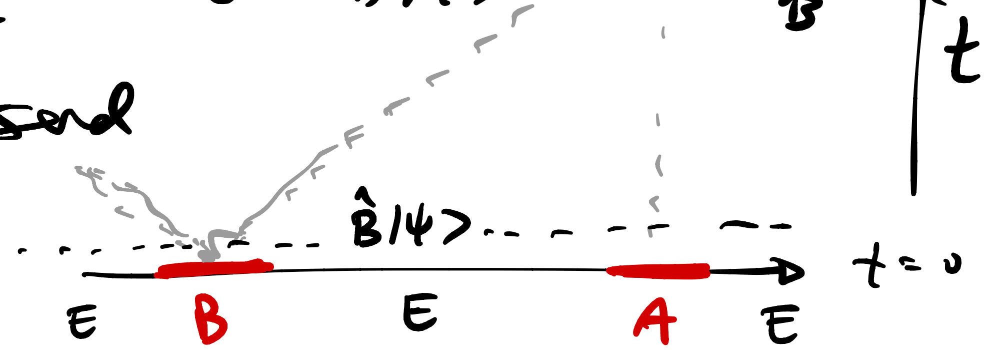
picks a time slice! $\quad$ ETCR.

3.1 Causality & antiparticles.

Causality principle : events shouldn't precede their causes.

Causality in QFT:

$$e^{-iHt}\hat{B}|\psi\rangle = |\psi(t)_B\rangle$$



$$|\psi\rangle = |\psi(t=0)\rangle$$

Suppose B wants to send a signal to A.

B acts w/ a unitary operator $\hat{B}$. (localized near B)

Then we wait. System evolves by some $H = H^\dagger$

A measures an operator $\hat{A}$. (localized near A)

$$\langle \hat{A}\rangle_B = \langle\psi(t)_B|\,\hat{A}\,|\psi(t)_B\rangle$$

$$= \langle\psi|\hat{B}^\dagger \underbrace{e^{iHt}\,\hat{A}\,e^{-iHt}}_{\hat{A}(t)} B|\psi\rangle$$

$$= \underbrace{\langle\psi|\hat{A}(t)|\psi\rangle - \langle\psi|\overbrace{B^\dagger B}^{=\mathbb{1}}\hat{A}(t)|\psi\rangle}_{=0} + \langle\psi|B^\dagger \hat{A}(t) B|\psi\rangle$$

$$= \underbrace{\langle\psi|\hat{A}(t)|\psi\rangle}_{\text{indep of what B did!}} - \langle\psi|B^\dagger \underbrace{[B, \hat{A}(t)]}_{\text{if } [B,A(t)]=0}|\psi\rangle$$

If $[B, A(t)] = 0$ NO SIGNAL CAN be received!

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w/o nonzero commutators entanglement doesn't help.

e.g.

$$|\psi\rangle = |\uparrow_A \downarrow_B\rangle - |\downarrow_A \uparrow_B\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$$

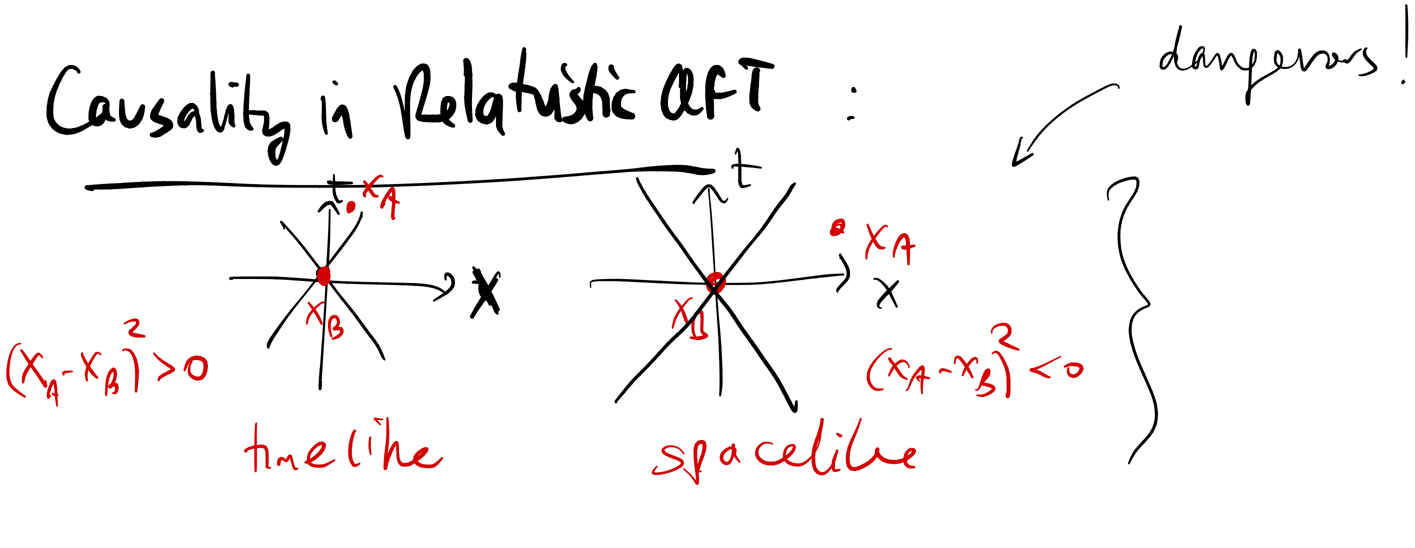
$$[\vec{S}_A \otimes \mathbb{1}_B, \mathbb{1}_A \otimes \vec{S}_B] = 0.$$

The groundstate of a quantum field has a lot of entanglement between spatial locations.

But $[\phi(\vec{x}), \phi(\vec{y})]_{ETCR} = 0$

$\Longrightarrow$ no signals.

Causality in Relativistic QFT:



A Lorentz inv't theory can only be causal if $\forall$ local operators A, B

$$[A(x_A), B(x_B)] = 0$$

$\forall\ x_A, x_B$ spacelike separated $(x_A - x_B)^2 < 0$.

In scalar QFT: Any local op is made from $\phi(x)$

$$\phi(x^\mu) = \int \frac{d^d p}{\sqrt{2\omega_p}} \left(a_{\vec p}\, e^{-i p_\mu x^\mu} + a_{\vec p}^\dagger\, e^{+i p_\mu x^\mu} \right)_{p^0 \equiv \omega_{\vec p}}$$

$$\equiv \underbrace{\phi^{(+)}(x)}_{\omega \geq 0} + \underbrace{\phi^{-}(x)}_{\omega \leq 0}.$$

$$[\phi^{(\pm)}(x), \phi^{(\pm)}(y)] = 0 \Leftarrow \begin{cases} [a, a] = 0 \\ [a^\dagger, a^\dagger] = 0 \end{cases}$$

$$[\phi(x), \phi(y)] = \int \frac{d^d p}{2\omega_{\vec p}} \left(e^{-i p_\mu (x-y)^\mu} - e^{i p_\mu (x-y)^\mu} \right)$$

$$= \int d^{d+1} p\ \underbrace{2\pi \delta(p^2 - m^2)\, \theta(p^0)}_{\text{Lorentz inv't}} \left(\quad \right)$$

Suppose $(x-y)^2 < 0$ (spacelike):

$\implies \exists$ a frame where x, y are at same time.

i.e. $\Lambda^\mu{}_\nu (x-y)^\nu = (\underset{\Delta t = 0}{0}, \Delta\vec{x})^\mu \equiv \tilde{x}^\mu$

$\exists \Lambda^\mu{}_\nu$

let $\tilde{p}^\mu = \Lambda^\mu{}_\nu p^\nu$ so that

$$p^\mu (x-y)_\mu = p^\mu \eta_{\mu\nu} (x-y)^\nu$$

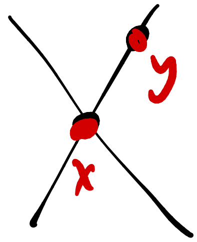
$$= \tilde{p}^\sigma \underbrace{(\Lambda^{-1})^\mu{}_\sigma \, \eta_{\mu\nu} \, (\Lambda^{-1})^\nu{}_\rho}_{= \eta_{\sigma\rho} \text{ (def. of Lorentz)}} \tilde{x}^\rho$$

$$= \tilde{p}^\sigma \tilde{x}_\sigma .$$

$$\implies [\phi(x), \phi(y)] \overset{(x-y)^2<0}{=} \int d^{d+1}p \; 2\pi \delta(\tilde{p}^2 - m^2)\, \theta(\tilde{p}^0) \times$$

$$\Big(\underbrace{e^{-i\tilde{\vec{p}}\cdot\Delta\vec{x}} - e^{+i\tilde{\vec{p}}\cdot\Delta\vec{x}}}_{\text{odd under } \tilde{\vec{p}} \to -\tilde{\vec{p}}} \Big) = 0$$

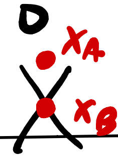
same arg. for $[\pi, \pi)$ and $[\phi, \pi]$



$(x-y)^2 = 0$

(if $\underline{\underline{(x-y)^2 < 0}}$ strict.)

* Note: if $\underline{(x-y)^2 > 0}$



$[\phi(x), \phi(y)] \neq 0.$

$$[\phi(x), \phi(y)] = \underbrace{[\phi^{(+)}(x), \phi^{(-)}(y)]}_{\equiv \hat{\Delta}^{+}(x-y)} + \underbrace{[\phi^{(-)}(x), \phi^{(+)}(y)]}_{\equiv \hat{\Delta}^{-}(x-y)}$$

$$= -\hat{\Delta}^{+}(y-x)$$

Because $[a, a^\dagger] \propto \mathbb{1}$

$\hat{\Delta}^{+} = \Delta^{+}$ is a c-number

$\propto \mathbb{1}$.

$\langle\psi|\psi\rangle = 1$

$\forall |\psi\rangle$

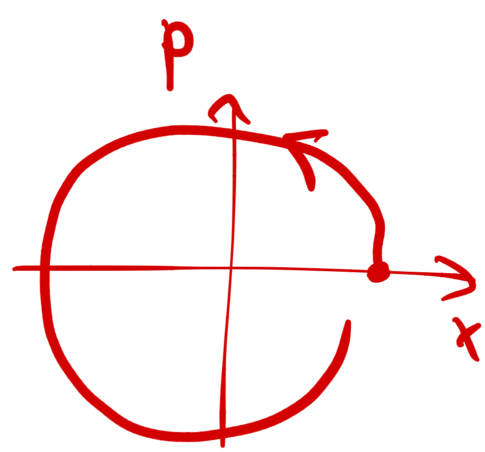
$$\Delta^{+}(x-y) = \langle\psi| \hat{\Delta}^{+}(x-y) |\psi\rangle$$

$$= \langle 0| \hat{\Delta}^{+}(x-y) |0\rangle$$

$$= \langle 0| [\phi^{(+)}(x), \phi^{(-)}(y)] |0\rangle$$

$\Leftarrow \phi^{(+)}|0\rangle = 0$

$\langle 0| \phi^{(-)} = 0$



$x(t) = \cos t \, x + \sin t \, p$

$[x(0), x(t)] \neq 0.$

$$\Delta^+(x-y) = \langle 0|\,\phi^{(+)}(x)\,\phi^{(-)}(y)\,|0\rangle - \underbrace{\langle 0|\phi^{(-)}}_{=0}(y)\underbrace{\phi^{(+)}(x)|0\rangle}_{=0}$$

$\langle 0|\phi^- $

$\phi^+|0\rangle$

$$= \underline{\langle 0|}\ \underline{\phi(x)}\ \underline{\phi(y)}\,\underline{|0\rangle}$$

is a <u>propagator</u>.

$$\Delta^+(x-y) = \sum_p \left(x \xleftarrow{\ p\ } y\right)$$

$$= \int \frac{\text{đ}^d p}{2\omega_p}\, e^{-ip^\mu (x-y)_\mu}\Big|_{p^0=\omega_p}$$

$$= \begin{cases} \underline{e^{-mr}}, & |x^0-y^0| \ll |\vec{x}-\vec{y}|,\quad (x-y)^2 = -r^2 \quad \text{very spacelike } r \gg 1/m \\ e^{-imt} & |x^0-y^0| \gg |\vec{x}-\vec{y}|,\quad (x-y)^2 = +t^2 \quad \text{very timelike} \end{cases}$$

<u>Note</u>: $\langle 0|\phi(x)\phi(y)|0\rangle \neq 0$ outside the lightcone $\left((x-y)^2 < 0\right)$

The cancellation in $[\Phi(x), \phi(y)] = 0$ for $(x-y)^2 < 0$

is interference betw. particles & antiparticles.

$$\Phi(x) = \Phi^{(+)}(x) + \Phi^{(-)}(x) \qquad \left(\Phi^* = \dots \right)$$

$$\Phi^{(+)}(x) = \int \frac{d^d p}{\sqrt{2\omega_p}} e^{-i p_\mu x^\mu} a_p \Big|_{p_0 = \omega_p}$$

$$\Phi^{(-)}(x) = \int \frac{d^d p}{\sqrt{2\omega_p}} e^{+ipx} b_p^\dagger \Big|$$

$$\Delta_a^+(x-y) \equiv \langle 0 | [\Phi^+(x), \Phi^{*-}(y)] | 0 \rangle$$

$$= \langle 0 | \Phi(x) \Phi^*(y) | 0 \rangle$$

$$-\Delta_b^-(y-x) \equiv \langle 0 | [\Phi^{*+}(y), \Phi^-(x)] | 0 \rangle$$

$$= \langle 0 | \Phi^*(y) \Phi(x) | 0 \rangle$$

$$\langle 0 | [\Phi(x), \Phi^*(y)] | 0 \rangle = \Delta_a^+(x-y) + \Delta_b^-(y-x)$$

$$= \sum_p \left[\left(\underset{x}{\bullet} \xleftarrow[p]{\text{particle}} \bullet\, y \right) - \left(\underset{x}{\bullet} \xrightarrow[p]{\text{antiparticle}} y \right) \right]$$

Q: what's the rel'n between

$\Delta^{\pm}(x-y)$ and $\langle 0|T\,\phi(x)\,\phi(y)|0\rangle \equiv \Delta_T(x-y)$

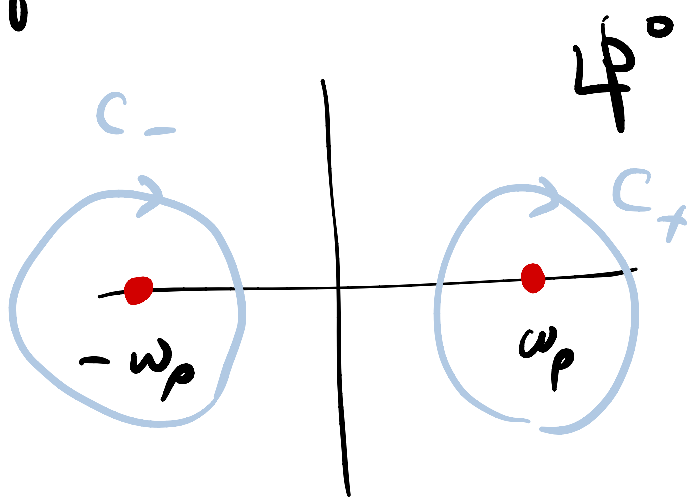
CLAIM: any of these is

$$\Delta(x) = \int_C d^{d+1}p \; \frac{i}{p^2-m^2}\, e^{-ip^\mu x_\mu}$$

for some choice of contour C.

poles at

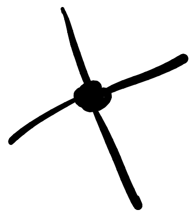
$$0 = p^2 - m^2 = p_0^2 - \vec{p}^{\,2} - m^2$$



$$\int_{C_+} d^{d+1}p \; \frac{e^{-ip^\mu x_\mu}\, i}{p^2-m^2} = \int \frac{d^dp}{2\omega_p}\, e^{-ip_\mu x^\mu}\Big|_{p_0=\omega_p}$$

$$= \Delta^+(x).$$

$$Z = \int dq\, e^{-S_0}\, \underbrace{e^{-Tq^4}}_{1 - Tq^4 + \frac{1}{2}(Tq^4)^2 + \dots}$$



$\langle q_a q_b \rangle =$  $+$ 

$$\sim \int dq\, e^{-S_0} q_a q_b \qquad \sim \int dq\, e^{-S_0} (-Tq^4)\, q_a q_b$$

$$\propto \delta_{\sum_i k_i}$$

$\uparrow$

legs going into the vertex.

$$q_a = \sum_k e^{ika} q_k$$

$$\langle q_{k_1}, q_{k_2} \rangle$$