$$\mathcal{L}_{\text{Maxwell}}[A_\mu] = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}(E^2-B^2)$$

choose: $\vec{\nabla}\cdot\vec{A}=0$ gauge, $A_0 = \Phi = 0.$

$$\pi_{A_i} = \frac{\partial \mathcal{L}_{max}}{\partial \dot{A}_i} = E^i = -\partial_t A_i$$

$$H = \frac{1}{2}\int d^3x\,(\vec{E}^2+\vec{B}^2)$$

$\vec{B}=\vec{\nabla}\times\vec{A}$

[no A^2 by gauge inv.]

compare: $H_{\text{phonons}} = \frac{1}{2}\int dx\,(\pi^2+(\partial_x\phi)^2)$

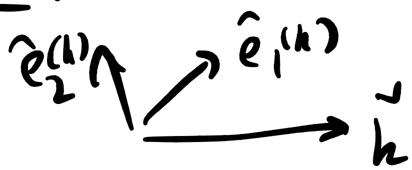
no ϕ^2 by $\phi\to\phi+\epsilon$ sym

$$\phi = \sum_k \sqrt{\frac{\hbar}{2\omega_k}}\,\underbrace{e^{i(kx-\omega_k t)}}_{\substack{\text{sol'n of}\\ \text{eom}\\ \omega_k^2=\vec{k}^2}}\,a_k + h.c.$$

Basis of sol'ns of Maxwell's eqns: $\left\{A_{k,s}(x,t) = \vec{e}_s(k)\,e^{i(\vec{k}\cdot\vec{x}-\omega_k t)}\right\}$

$$\vec{\nabla}\cdot\vec{A}=0 \Longrightarrow \underbrace{\hat{k}\cdot\vec{e}_s(\hat{k})=0}.$$

s=1,2.



$$\vec{A}(\vec{r},t) = \int d^3k \, \frac{1}{\sqrt{2\omega_k}} \sum_{s=1,2} \Big(\underbrace{a_{\vec{k},s} \, \vec{e}_s(\hat{k}) \, e^{i(\vec{k}\cdot\vec{r} - \omega_k t)}}_{\geq 0 \text{ energy sol'ns}} + \underbrace{a^{\dagger}_{\vec{k},s} \, \vec{e}^{\,*}_s(\hat{k}) \, e^{-i\vec{k}\cdot\vec{r} + i\omega_k t}}_{\leq 0 \text{ energy sol'ns}} \Big)$$

$$\vec{k}\cdot\vec{r} - \omega_k t \equiv -kx$$

$$\omega_k = \sqrt{\vec{k}\cdot\vec{k}} = \sqrt{k^2} \geq 0$$

$$\underline{U(t) = e^{-iHt}}$$

$$[a_{k,s}, a^{\dagger}_{k',s'}] = (2\pi)^d \delta^d(k-k') \delta_{ss'}$$

we can choose a basis of e's:

$$\forall \vec{k} \quad \sum_{s=1,2} e_{si}(\hat{k}) e^*_{sj}(\hat{k}) = \underbrace{\delta_{ij} - \hat{k}_i \hat{k}_j}$$

rank 1

annihilates $\vec{k}$

$$(\delta_{ij} - \hat{k}_i \hat{k}_j) k_j = 0$$

$$\vec{E} = -\partial_t \vec{A} =$$

$$= i \int d^3k \sum_s \sqrt{\frac{\hbar \omega_k}{2}} \Big(a_{ks} e_s(k) e^{-ikx} - a^{\dagger}_{ks} e^*_s(k) e^{+ikx} \Big)$$

$$\vec{B} = \vec{\nabla} \times \vec{A} = \int d^3k \sum_s \sqrt{\frac{\hbar}{2\omega_k}} \, i\vec{k} \times \Big(a_{ks} e_s(k) e^{-ikx} - a^{\dagger}_{ks} e^*_s e^{+ikx} \Big)$$

Recall
$[\phi(x), \pi(y)] = i\delta(x-y) \Longleftrightarrow [a_k, a^\dagger_{k'}] = 2\pi\delta(k-k')$

$$[A_i(x,t=0), E_j(y,t=0)] = -i\hbar \int d^3k\, e^{-ik\cdot(x-y)} (\delta_{ij} - \hat{k}_i\hat{k}_j)$$

only 2 dof.

check: $\partial_t^2 \vec{A} = -\partial_t \vec{E} = -\frac{i}{\hbar}[H, \vec{E}] = \vec{\nabla}^2 A$

$$\rightarrow H = \int d^3k\, \hbar\omega_k \left(a^\dagger_{ks} a_{ks} + \frac{1}{2}\right).$$

energy of vac:

$$E_0 = \frac{1}{2}\sum_{ks} \hbar\omega_k = 2^3 \int d^3k\, \hbar c k$$

$= \infty$ in 2 ways ① IR. $E_0 = L^3 \varepsilon \leftarrow$ energy density.

② UV $\varepsilon \propto \int d^3k\, k \sim \Lambda^4$?

$\Lambda \sim \frac{2\pi}{a}$ max wavef'n.

1.4 Lagrangian F.T. & Symmetries.

dofs $\{\phi_r(x)\}$ $\quad r = 1 \ldots \#$ of components

assume $S[\phi] = \int d^{d+1}x \, \mathcal{L}(\phi, \partial_\mu \phi)$ ←

$\underline{\text{g.}}$ $\begin{cases} \mathcal{L}_{KG} = \underline{\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi} - \frac{1}{2}m^2\phi^2 \\ \mathcal{L}_{max} = -\frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2e^2}(\underline{E}^2 - \underline{B}^2) \end{cases}$

A word about units & dim'l analysis

when $\hbar = c = 1$ everything is a mass$^\#$

$[\mathcal{O}] = \#$ if $\mathcal{O}$ is a mass$^\#$.

$[p_\mu] = [\hbar k_\mu] = [(\hbar\omega, \hbar\vec{k})_\mu] = 1$

$[x^\mu = (t, \vec{x})^\mu] = -1 \Rightarrow [\frac{\partial}{\partial x^\mu}] = +1$.

$[S] = 0 \qquad e^{iS/\hbar}$

$[\mathcal{L}] = d+1 \qquad [A_\mu] = 1 \qquad \underline{[e] = 0}.$

$[\phi] = \frac{d-1}{2} \qquad [E] = [B] = 2$

eom: $$0 = \frac{\delta S}{\delta \phi_r(x)}$$ ← more robust.

$\overset{\Longrightarrow}{\text{if}}$ $$0 = \frac{\partial \mathcal{L}}{\partial \phi_r} - \partial^\mu \frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi_r)}$$

$\mathcal{L} = \mathcal{L}(\phi, \partial_\mu \phi)$

and no boundary

of spacetime

$$S[\phi] = \int_M \frac{(\partial\phi)^2 - \phi^2}{2}$$

$$\frac{\delta S}{\delta \phi(x)} = \int_M \partial_\mu \phi \, \underline{\underline{\partial^\mu \delta(x-y)}} - \phi \, \delta(x-y)$$

$$\overset{\text{IBP}}{=} \left(-\partial_\mu \partial^\mu \phi - \phi\right)$$

$$+ \underbrace{\int_{\partial M} \partial_\mu \phi \, \delta(x-y) \cdot d\eta^\mu}_{\text{ignore.}}$$

who is ϕ?

$$Z = \int [D\phi] \, e^{iS[\phi]/\hbar}$$

an integration variable.

eg: $\phi = \frac{1}{D}\left(\chi - B/c\right)$

$$\longrightarrow \mathcal{L} = A + B\chi + \frac{1}{2} C\chi^2 + \frac{1}{2} D (\partial\chi)^2 + \dots$$

eg: a $\mathbb{Z}_2$ symmetry $S[\phi] = S[-\phi]$ $\Longrightarrow$ no odd powers of ϕ in $\mathcal{L}$.

$\phi \to -\phi$.

$\mathcal{L}_{KG}$ is special in that

→ ① $\phi \to -\phi$ symmetry.

→ ② no interactions!

→ ③ no higher deriv. terms.

Noether's Thm & Continuous Symmetries

Suppose $S[\phi'(x)] = S[\phi(x)]$ ← does not depend on x.

(i.e. $\phi \to \phi'$ is a symmetry.)

is continuous means

$\phi(x) \longmapsto \phi(x) + \epsilon \Delta\phi(x)$ is a sym for $\epsilon \ll 1$.

2

let $\delta_\epsilon S[\phi] \equiv S[\phi + \epsilon\Delta\phi] - S[\phi]$.

made of ϕ's

step 1: if $\delta_\epsilon S[\phi] = 0$ for ϵ constant

then $\delta_{\epsilon(x,t)} S[\phi] =: \int \partial_\mu \epsilon(x) j^\mu(x) + O(\epsilon^2)$

step 2: if $\underline{\phi}$ solves eom

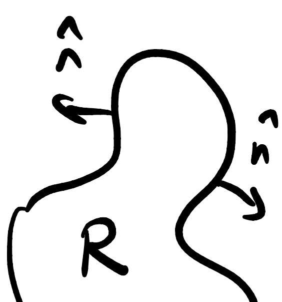
$$0 = \frac{\delta S}{\delta \phi}\Big|_{\phi=\underline{\phi}} \qquad \text{i.e.} \qquad \delta S\Big|_{\phi=\underline{\phi}} = 0 .$$

$$0 \overset{\text{eom}}{=} d_{\epsilon(x)} S\Big|_{\underline{\phi}} = S[\phi(x) + \epsilon(x)\Delta\phi(x)] - S[\phi(x)]\Big|_{\underline{\phi}}$$

$$\overset{\text{def of } j}{=} \int d^{d+1}x\, \partial_\mu \epsilon(x)\, j^\mu \overset{\text{IBP}}{=}$$

$$- \int d^{d+1}x\, \epsilon(x)\, \underline{\partial_\mu j^\mu(x)} . \qquad \forall \epsilon(x)$$

$$\Longrightarrow \boxed{\partial_\mu j^\mu(x)\Big|_{\underline{\phi}} = 0 .}$$



fixed in time region of space

$$Q_R \equiv \int_R d^d x\, j^0$$

$$\partial_t Q_R = \int_R d^d x\, \partial_t j^0 = - \int_R d^d x\, \vec{\nabla} \cdot \vec{j}$$

$$\overset{\text{Stokes'}}{=} \int_{\partial R} d^{d-1}x\, \hat{n} \cdot \vec{j} .$$

More prosaic version if $\mathcal{L} = \mathcal{L}(\phi, \partial_\mu \phi)$:

$$\mathcal{L}(\phi', \partial_\mu \phi') \overset{\text{sym}}{=} \mathcal{L}(\phi, \partial_\mu \phi) + \epsilon \, \underline{\underline{\partial_\mu J^\mu}} \qquad \boxed{\text{step } 1'}$$

step 2' : $$\mathcal{L}(\phi', \partial_\mu \phi') \overset{\text{calculus}}{=} \mathcal{L}(\phi, \partial_\mu \phi) + \epsilon \left(\frac{\partial \mathcal{L}}{\partial \phi} \Delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\mu \Delta\phi \right)$$

$$\overset{\text{IBP}}{=} \mathcal{L}(\phi, \partial_\mu \phi) + \epsilon \cancel{\left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right)}^{\,0} \Delta\phi + \epsilon \, \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta\phi \right)$$

$$\Longleftrightarrow \quad j^\mu = \sum_r \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_r)} \Delta\phi_r - J^\mu .$$

is conserved $\quad \partial_\mu j^\mu = 0 .$

Noether : continuous sym $\Longrightarrow$ conserved current

$\Longleftarrow$

Noether converse: suppose given a conserved charge $Q = Q^\dagger$ w/ $[H, Q] = 0$.

let $\delta\phi(x) \equiv i\epsilon\,[Q, \phi(x)]$.

"Q generates the symmetry $\delta\phi$".

finite transf: $\phi \to \phi' \equiv e^{i\epsilon Q}\,\phi\, e^{-i\epsilon Q}$.

$U \equiv e^{i\epsilon Q}$ is unitary. $U^\dagger U = UU^\dagger = \mathbb{1}$.

$[H, U] = 0 \Leftarrow [Q, H] = 0$.

[note: $\frac{\partial}{\partial\epsilon}\phi' = i\,[Q, \phi]$

examples: ① Suppose $\mathcal{L}(\phi, \partial_\mu\phi) = \mathcal{L}(\partial_\mu\phi)$.

eg: $S[\phi] = \frac{1}{2}\int \partial_\mu\phi\,\partial^\mu\phi$.

$\Rightarrow$ shift symmetry $\phi \mapsto \phi' = \phi + \epsilon$.

let $\epsilon = \epsilon(x)$

$$\delta_{\epsilon(x)} S[\phi] = S[\phi + \epsilon(x)] - S[\phi]$$

$$\overset{\epsilon \ll 1}{=} \int \epsilon\, \partial_\mu \partial^\mu \phi$$

$\Longrightarrow j^\mu = \partial^\mu \phi$. $\quad \left(\partial_\mu j^\mu = 0 \ \underline{\underline{\text{is}}} \text{ the eom} \right)$.

$$Q = \int_{\text{space}} j^0 = \int_{\text{space}} \phi'$$

generates $\quad \delta\phi(x) = \phi' - \phi = i\epsilon [Q, \phi]$

$$= i\epsilon \left[\int d^d y\, \pi(y), \phi(x) \right]$$

$\left([\phi(x), \pi(y)] = i\delta(x-y) \right)$

$$= i\epsilon \int d^d y \left(-i\, \delta^d(x-y) \right)$$

$$= \epsilon \quad \checkmark .$$

② $S[\Phi, \Phi^*]$ inv't under $\Phi \to e^{i\epsilon} \Phi$

$$= \Phi + i\epsilon\Phi + O(\epsilon^2)$$

$$\delta\Phi = i\Phi .$$

$$\to Q = \int d^d x\, j^0 = \int đ^d k \left(a_k^\dagger a_k - b_k^\dagger b_k \right).$$

where $\begin{cases} \Phi(x) = \int d^d k \sqrt{\frac{1}{2\omega_k}} \left(a_k e^{-ikx} + b_k^\dagger e^{ikx} \right) \\ \Phi^\dagger(x) = \int d k \sqrt{\ } \left(b e^{-ikx} + a^\dagger e^{ikx} \right) \end{cases}$

a & b are antiparticles of ea. other.

(3) Spacetime translation. " $x^\mu \to x^\mu - a^\mu$ ".

acts on fields by

for a small

Taylor

$$\phi(x) \mapsto \phi'(x) = \phi(x+a) = \phi(x) + a^\nu \partial_\nu \phi$$

d+1 transformations $\Delta_\nu \phi = \partial_\nu \phi$. $+O(a^2)$

is a sym if $\mathcal{L}$ depends on x only

through ϕ, $\partial_\mu \phi$, $\partial_\nu \mathcal{L} = 0$.

$$\left(0 \neq \frac{d}{dx^\nu} \mathcal{L} = \partial_\nu \phi \frac{\partial \mathcal{L}}{\partial \phi} + \partial_\nu \partial_\mu \phi \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \right.$$

$\to$ d+1 conserved currents = energy-momentum tensor.

$$\mathcal{L}(\phi(x), \partial_\mu\phi(x)) \mapsto \mathcal{L}(\phi(x+a), \partial_\mu\phi(x+a)) \overset{\text{Taylor}}{=} \mathcal{L}(\phi(x), \partial_\mu\phi(x)) + a^\mu \frac{d}{dx^\mu}\mathcal{L} + O(a^2)$$

$$= \mathcal{L} + a^\nu \partial_\mu \big(\underbrace{\delta^\mu_\nu \mathcal{L}}_{J^\mu{}_\nu}\big)$$

prosaic method

$$\Rightarrow T^\mu{}_\nu \overset{\downarrow}{=} \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)} \Delta_\nu \phi - J^\mu{}_\nu$$

$$= \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)} \partial_\nu \phi - \delta^\mu_\nu \mathcal{L} .$$

<u>eg</u>: <u>$\nu = 0$</u>: $T^0{}_0 = \pi \dot{\phi} - \mathcal{L} = h$.

energy density.

<u>eg</u>: $\nu = i$

$$P_i = \int d^d x \, T_i{}^0 \overset{\substack{\text{KG} \\ \text{scalar}}}{=} - \int d^d x \, \pi \, \partial_i \phi .$$

$$= \int đ^d k \, k_i \, a_k^\dagger a_k .$$

= momentum: $a^\dagger_{k_2} a^\dagger_{k_1} |0\rangle$ has momentum $\vec{k}_1 + \vec{k}_2$.

$$\phi_{\alpha=1,2} = \sum_k \frac{1}{\sqrt{2\omega_k}} \left(a_\alpha e^{-ikx} + a^\dagger_\alpha e^{ikx} \right)$$

$$H = \int_{\text{space}} \sum_\alpha \left(\pi_\alpha^2 + \phi_\alpha^2 \right) = \int_{\text{space}} \left(\Pi^\dagger \Pi + \Phi^\dagger \Phi \right)$$

$$\Phi \equiv \phi_1 + i\phi_2 .$$

$$= \sum_k \frac{1}{\sqrt{2\omega_k}} \left(a e^{-ikx} + b^\dagger e^{ikx} \right)$$

$$\begin{cases} a \equiv a_1 + i a_2 \\ b \equiv a_1 - i a_2 \end{cases}$$

(Beware $\sqrt{2}$)

$$[a_{\alpha k}, a^\dagger_{\beta k'}] = \delta_{\alpha\beta}\, \delta(k-k') \Longrightarrow \begin{cases} [a, a^\dagger] = \delta \\ [a, b^\dagger] = 0 \\ [b, b^\dagger] = \delta . \end{cases}$$

$$\begin{cases} [\phi(x), \pi(y)] = i\delta(x-y) \\ [\phi(x), \phi(y)] = 0 \quad \forall x, y \end{cases}$$

$$\frac{\partial}{\partial x} (\text{BHS}) \Longrightarrow [\partial_x \phi(x), \phi(y)] = 0 .$$

if $\underset{\text{smooth}}{\exists} A_\mu$ s.t. $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

$$\left(\partial_\rho \partial_\sigma A_\mu = \partial_\sigma \partial_\rho A_\mu \right)$$

then $\epsilon^{\mu\nu\rho\sigma} \partial_\nu F_{\rho\sigma} = 0.$

$$\pi_{A_0} = \frac{\partial \mathcal{L}_{\text{max}}}{\partial \dot{A}_0} = 0.$$

$$Z = \int DA_\mu \, e^{i \int \# A_0 (\vec{\nabla} \cdot \vec{E} - \rho) + \dots}$$

$$\int dx \, e^{ixp} = \delta(p)$$

$$\propto \delta[\vec{\nabla} \cdot \vec{E} - \rho]$$

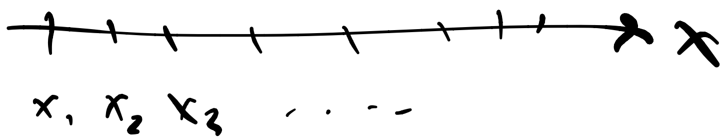
$\delta \phi = i\epsilon [(\vec{\nabla} \cdot \vec{E} - \rho), \phi]$ is the gauge variation.

if $\rho = 0$, $\Phi = 0$ $\rightsquigarrow$ $\vec{\nabla} \cdot \dot{\vec{A}} = 0.$

$$0 = \frac{\delta S[\phi]}{\delta \phi(x)}$$

$$\underline{\underline{\frac{\delta \phi(y)}{\delta \phi(x)} = \delta(x-y)}}$$

$$S[\phi] = S\left(\phi_1 \equiv \phi(x_1),\ \overset{\phi_2 \equiv}{\phi(x_2)},\ \phi(x_3)\ \dots\right)$$



$$\frac{\delta S}{\delta \phi(x_i)} = \frac{\partial S}{\partial \phi_i}$$

$$\underline{\underline{\frac{\partial \phi_i}{\partial \phi_j} = \delta_{ij}}}$$

$$T_{\mu\nu} \overset{?}{=} T_{\nu\mu}$$

$$\underline{A_1^{\mu\alpha} A_{2\,\alpha\nu}}$$

$$T^{\mu}{}_{\nu} = F^{\mu\alpha} F_{\alpha\nu} + \frac{1}{4} F_{\beta\gamma} F^{\beta\gamma} \delta^{\mu}_{\nu}$$

$$T_{\mu\nu} = \underline{\underline{F_{\mu}}}{}^{\alpha} F_{\alpha\nu} + \dots$$

$$T_{\nu\mu} = \underline{\underline{F_{\nu}}}{}^{\alpha}\, \underline{\underline{F_{\alpha\mu}}} + \dots = F_{\nu\alpha}\, F^{\alpha}{}_{\mu} + \dots$$

$$= F_{\nu\beta}\, F_{\alpha\mu}\, \underbrace{\eta^{\beta\alpha}}_{=\,\eta^{\alpha\beta}}$$