University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215A QFT Fall 2021
Assignment 7

Due 11:59pm Thursday, November 11, 2021

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1. **Brain-warmers.**

   Consider the field theory with action

$$S[\phi] = \int d^{d+1}x \, \left(\frac{1}{2}(\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2) - \frac{g}{3!}\phi^3 \right).$$

   (a) State the Feynman rules in position space.

   (b) Draw the diagrams that correct the position-space two-point function at order g^2.

   (c) Find the symmetry factor for these diagrams.

2. **Particle creation by an external source.**

   Compare this problem with problem 4 on HW06.

   Consider the Hamiltonian

$$H = H_0 + \int d^3x \, (-j(t, \vec{x})\phi(x))$$

   where H_0 is the free Klein-Gordon Hamiltonian, ϕ is the Klein-Gordon field, and j is a c-number scalar function.

   (a) Show that the probability that the source creates *no* particles is given by

$$P(0) = \left| \langle 0 | \mathcal{T}\{ e^{+\mathbf{i}\int d^4x j(x)\phi_I(x)} \} |0\rangle \right|^2.$$

   (b) Evaluate the term in $P(0)$ of order j^2, and show that $P(0) = 1 - \lambda + \mathcal{O}(j^4)$ where

$$\lambda = \int \frac{đ^3 p}{2E_p} |\tilde{j}(p)|^2.$$

   We will show below that $\lambda = \langle N \rangle$ is the mean number of particles created by the source.

   (c) Represent the term computed in part 2b as a Feynman diagram. Now represent the whole perturbation series for $P(0)$ in terms of Feynman diagrams. (Hint: you have done this calculation already.) Show that this series exponentiates, so that it can be summed exactly $P(0) = e^{-\lambda}$.

(d) On the next problem set, after learning about the notion of final-state phase space, we'll find the probability for the source to create any number of particles.

3. **Propagator corrections in a solvable field theory.**

Consider a theory of a scalar field in D dimensions with action

$$S = S_0 + S_1$$

where

$$S_0 = \int d^D x \frac{1}{2} \left(\partial_\mu \phi \partial^\mu \phi - m_0^2 \phi^2 \right)$$

and

$$S_1 = -\int d^D x \frac{1}{2} \delta m^2 \phi^2 \ .$$

We have artificially decomposed the mass term into two parts. We will do perturbation theory in small δm^2, treating S_1 as an 'interaction' term. We wish to show that the organization of perturbation theory that we've seen lecture will correctly reassemble the mass term.

(a) Write down all the Feynman rules for this perturbation theory.

(b) Determine the 1PI two-point function in this model, defined by

$$\mathbf{i}\Sigma \equiv \sum (\text{all 1PI diagrams with two nubbins}) \, .$$

(c) Show that the (geometric) summation of the propagator corrections correctly produces the propagator that you would have used had we not split up $m_0^2 + \delta m^2$.

4. **Wick's theorem from Schwinger-Dyson equations.** [Bonus problem] Study the derivation of Wick's theorem from the Schwinger-Dyson equation for the n-point function of a free scalar field on page 81 of Schwartz' book.

5. **A background field.** [This is a bonus problem.]

Consider the following action for a real scalar field Φ:

$$S[\Phi] = \int d^{d+1} x \frac{1}{2} \left(\partial_\mu \Phi \partial^\mu \Phi - m^2 \Phi^2 - g\phi(x) \Phi^2 \right) .$$

The last term here is a cubic coupling between ϕ and Φ. But here we will treat $\phi(x)$ as a fixed background field (analogous to $j(x)$ on previous problems) which acts as a spacetime-dependent mass for the dynamical field Φ.

[I moved around some factors of **i** below.]

(a) Show that the two-point Green's function, $G(x,y) \equiv \langle\Omega|\mathcal{T}\Phi(x)\Phi(y)|\Omega\rangle$, satisfies the Schwinger-Dyson equation

$$-\mathbf{i}\delta^{d+1}(x-y) = \left(\partial^2 + m^2 + g\phi(x)\right) G(x,y). \tag{1}$$

(b) We would like to solve this differential equation. As a warmup, consider the case $g = 0$. Here is a trick: add a fictitious additional time direction T

$$\left(\partial_T - \left(\partial^2 + m^2\right)\right) G(x,y,T) = \mathbf{i}\delta^{d+1}(x-y)\delta(T) \tag{2}$$

This is just a diffusion equation (in $d+2$ dimensions and with a funny factor of $\mathbf{i}$!). Show that given a solution to (2), you can find the solution of (1) with $g = 0$ by

$$G(x,y) = \int_0^\infty dT G(x,y,T). \tag{3}$$

(c) Show that the solution to the diffusion equation (2) is

$$G(x,y,T) = \frac{\mathbf{i}}{(2\pi T)^\alpha} e^{a\frac{(x-y)^2}{2T} + b\frac{m^2}{2}T}. \tag{4}$$

Find α, a, b. Use this to construct the path integral representation

$$G(x,y,T) = \int_{x(0)=x}^{x(T)=y} [Dx] e^{-\mathbf{i}\int_0^T d\tau\left(\dot{x}^\mu\dot{x}_\mu + m^2\right)}.$$

(d) For the case of constant m^2, the infinitesimal solution (4) actually works for finite T. Show by differentiation that plugging (4) into (3) gives an integral representation of the free Klein-Gordon propagator.

(e) Now let $g \neq 0$ and suppose that ϕ is slowly varying. Generalize the path integral representation to include the dependence on ϕ.

(f) Consider a non-relativistic situation, where the spacetime points x and y are separated by a timelike distance large compared to $1/m$. Justify and use stationary-phase methods to show that the dominant contribution to the path integral is a straight-line trajectory between the two points x and y. Evaluate the resulting amplitude as a functional of $\phi(x)$.

This calculation shows that the heavy particle made by the field Φ can be treated as a source for ϕ propagating on a fixed path in spacetime.

(g) Redo the problem for a *charged* scalar field, Φ in the background of a vector potential A_μ, with

$$S[\Phi] = \int d^{d+1}x \frac{1}{2}\left(D_\mu\Phi^\star D^\mu\Phi - m^2\Phi^\star\Phi\right), \quad D_\mu\Phi \equiv \partial_\mu\Phi - \mathbf{i}A_\mu\Phi.$$

It will help to recall that the action of a classical charged particle is $\int d\tau\left(\dot{x}^2 + \dot{x}^\mu A_\mu(x)\right)$.

6. If you didn't finish it earlier, try again problem 5 on HW05 (the one about Catalan numbers). I'll post my solution with this problem set.