University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215A QFT Fall 2021
Assignment 6

Due 11:59pm Thursday, November 4, 2021

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1. **Recovering non-relativistic quantum mechanics.**

   Consider a complex scalar field, in the non-relativistic limit,

   $$\Phi = \sqrt{2m}e^{-\mathbf{i}mt}\Psi, \quad |\dot{\Psi}| \ll m\Psi.$$

   Recall that in this limit, the antiparticles disappear and the mode expansion is

   $$\Psi(x) = \int đ^d p\ e^{+\mathbf{i}\vec{p}\cdot\vec{x}}\mathbf{a}_p, \quad \Psi^\dagger(x) = \int đ^d p\ e^{-\mathbf{i}\vec{p}\cdot\vec{x}}\mathbf{a}_p^\dagger\ .$$

   (a) Show that

   $$\hat{P}_i \equiv \int đ^d p p_i \mathbf{a}_p^\dagger \mathbf{a}_p$$

   is the generator of translations and commutes with the Hamiltonian.

   (b) Let

   $$\hat{X}^i \equiv \int d^d x \Psi^\dagger(x) x^i \Psi(x).$$

   A state of one particle at location $\vec{x}$ is

   $$|x\rangle = \Psi^\dagger(x)|0\rangle\ .$$

   Show that

   $$\hat{X}^i|x\rangle = x^i|x\rangle\ .$$

   (c) Consider the general one-particle state

   $$|\psi\rangle = \int d^d x\ \psi(x)\Psi^\dagger(x)|0\rangle = \int d^x x\ \psi(x)|x\rangle\ .$$

   Show that

   $$\hat{X}^i|\psi\rangle = \int d^d x x^i \psi(x)|x\rangle$$

   and (a little more involved)

   $$\hat{P}^i|\psi\rangle = \int d^d x \left(-\mathbf{i}\frac{\partial}{\partial x^i}\psi(x)\right)|x\rangle\ ,$$

   which is the usual action of these operators on single-particle wavefunctions $\psi(x)$.

2. **Scalar Yukawa amplitudes.**

   Consider again the scalar Yukawa theory of a complex scalar Φ and a real scalar ϕ. In the following, assume all particles are in momentum eigenstates. Use artisanal methods.

   (a) Compute the amplitude for the annihilation of a Φ particle and a $\Phi^\star$ particle into a ϕ particle, at leading order in the coupling g.

   (b) Compute the amplitude for $\Phi + \phi \to \Phi + \phi$ scattering to the leading order in the coupling at which it is nonzero.

3. **Wick example.**

   For a real scalar field, verify by hand Wick's prediction for the difference

$$\mathcal{T}\left(\phi(x_1)\phi(x_2)\phi(x_3)\right) - :\phi(x_1)\phi(x_2)\phi(x_3):$$

4. **Fields and forces.** [from Banks] Consider a real free relativistic scalar field of mass m $S[\phi] = \int d^{d+1}x \frac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2\right)$.

   (a) Calculate the vacuum expectation value

$$\langle 0| \mathcal{T}\left(e^{\mathbf{i}\int d^{d+1}x\ \phi(x)J(x)}\right)|0\rangle \equiv e^{\mathbf{i}W[J]}$$

   where J is a fixed, external source. Use Wick's theorem. Make a series expansion in powers of J and draw some diagrams. To understand the structure of the series, recall the formula on a previous homework for $\left\langle e^{K\cdot q}\right\rangle$ in any gaussian theory.

   (b) Now specialize to the case where the source is static and is present for a time $2T$:

$$J(x) = J_{\text{static}} \equiv \theta(T-t)\theta(t+T)\left(\delta^d(x) - \delta^d(x-R)\right)$$

   with $T \gg R \gg 1/m$. You should find an answer of the form

$$W\left[J_{\text{static}}(x)\right] = TV(R)$$

   where $V(R)$ is the Yukawa potential.

   (c) Chant the following incantation:

   Static sources experience a force due to exchange of virtual particles.

   Feel happy at having reproduced by canonical methods the answer we found earlier using path integral methods.