University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215A QFT Fall 2021
Assignment 3

Due 11:59pm Thursday, October 14, 2021

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1. **Maxwell's equations, quantumly.**

   (a) Check that the oscillator algebra for the photon creation and annihilation operators
   $$[\mathbf{a}_{ks}, \mathbf{a}^\dagger_{k's}] = \delta^3(k-k')\delta_{ss'}. \qquad (1)$$
   implies (using the mode expansion for $\mathbf{A}$) that
   $$[\mathbf{A}_i(\vec{r}), \mathbf{E}_j(\vec{r}')] = -\mathbf{i}\hbar \int đ^3k\ e^{\mathbf{i}\vec{k}\cdot(\vec{r}-\vec{r}')} \left(\delta_{ij} - \hat{k}_i\hat{k}_j\right)$$
   (and also $[\mathbf{A}_i(\vec{r}), \mathbf{A}_j(\vec{r}')] = 0$ and $[\mathbf{E}_i(\vec{r}), \mathbf{E}_j(\vec{r}')] = 0$).
   Conclude that it's not possible to simultaneously measure $E_x(\vec{r})$ and $B_y(\vec{r})$.

   (b) Using the result of the previous part, check that the wave equation for $\mathbf{A}_i(x)$ follows from the Heisenberg equations of motion
   $$+\partial_t\vec{\mathbf{E}} = \frac{\mathbf{i}}{\hbar}[\mathbf{H}, \vec{\mathbf{E}}].$$

2. **Goldstone boson.** Here is a simple example of the Goldstone phenomenon, which I mentioned briefly in lecture. Consider again the complex scalar field from a previous assignment.

   Suppose the potential is
   $$V(\Phi^\star\Phi) = g\left(\Phi^\star\Phi - v^2\right)^2$$
   where g, v are constants. The important features of V are that (1) it is only a function of $|\Phi|^2 = \Phi\Phi^\star$, so that it preserves the particle-number symmetry generated by $\mathbf{q}$ which was the hero a previous homework problem, and (2) the minimum of $V(x)$ away from $x = 0$.

   Treat the system classically. Write the action $S[\Phi, \Phi^\star]$ in polar coordinates in field space:
   $$\Phi(x,t) = \rho e^{\mathbf{i}\theta}$$
   where both ρ, θ are functions of space and time.

(a) Consider constant field configurations, and show that minimizing the potential fixes ρ but not the phase θ.

(b) Compute the mass^2 of the ρ field about its minimum, $m_\rho^2 = \frac{1}{2}\partial_\rho^2 V|_{\rho=v}$.

(c) Now ignore the deviations of ρ from its minimum (it's heavy and slow and hard to excite), but continue to treat θ as a field. Plug the resulting expression

$$\Phi = ve^{\mathbf{i}\theta(x,t)}$$

into the action. Show that θ is a massless scalar field.

(d) How does the $U(1)$ symmetry generated by $\mathbf{q}$ act on θ?

3. **Gaussian integrals are your friend.**

(a) Show that

$$\int_{-\infty}^{\infty} dx e^{-\frac{1}{2}ax^2+jx} = \sqrt{\frac{2\pi}{a}} e^{\frac{j^2}{2a}}.$$

[Hint: square the integral and use polar coordinates.]

(b) Consider a collection of variables $x_i, i = 1..N$ and a real, symmetric matrix a_{ij}. Show that

$$\int \prod_{i=1}^{N} dx_i e^{-\frac{1}{2}x_i a_{ij} x_j + J^i x_i} = \frac{(2\pi)^{N/2}}{\sqrt{\det a}} e^{\frac{1}{2}J^i a_{ij}^{-1} J^j}.$$

(Summation convention in effect, as always.)

[Hint: change integration variables to diagonalize a. $\det a = \prod a_i$, where a_i are the eigenvalues of a.]

(c) I include this problem partly because it might be helpful for a future problem. In that regard, for any function of the N variables, $f(x)$, let

$$\langle f(x) \rangle \equiv \frac{\int \prod_{i=1}^{N} dx_i e^{-\frac{1}{2}x_i a_{ij} x_j} f(x)}{Z[J=0]}, \quad Z[J] = \int \prod_{i=1}^{N} dx_i e^{-\frac{1}{2}x_i a_{ij} x_j + J^i x_i}.$$

Show that

$$\langle x_i x_j \rangle = \partial_{J_i} \partial_{J_j} \log Z[J]|_{J=0} = a_{ij}^{-1}$$

Also, convince yourself that

$$\left\langle e^{J_i x_i} \right\rangle = \frac{Z[J]}{Z[J=0]}.$$

(d) Note that the number N in the previous parts may be infinite. This is really the only path integral we know how to do.

4. **Gaussian identity.** Show that for a gaussian quantum system

$$\left\langle e^{\mathbf{i}K\mathbf{q}} \right\rangle = e^{-A(K)\left\langle \mathbf{q}^2 \right\rangle}$$

and determine $A(K)$. Here $\langle ... \rangle \equiv \langle 0| ... |0\rangle$, vacuum expectation value. Here by 'gaussian' I mean that $\mathbf{H}$ contains only quadratic and linear terms in both $\mathbf{q}$ and its conjugate variable $\mathbf{p}$ (but for the formula to be exactly correct as stated you must assume $\mathbf{H}$ contains only terms quadratic in $\mathbf{q}$ and $\mathbf{p}$; for further entertainment fix the formula for the case with linear terms in $\mathbf{H}$).

I recommend using the path integral representation (with hints from the previous problem). Alternatively, you can use the harmonic oscillator operator algebra. Or, even better, do it both ways.

5. **Casimir energy from balls and springs.** [I am declaring this a bonus problem because it is quite a bit more involved than I initially thought.] Regularize the Casimir energy of a 1d scalar field by discretizing space. That is, suppose there are $N \equiv d/a \in \mathbb{Z}$ lattice points in the left cavity:

$$| \leftarrow d \rightarrow | \longleftarrow \qquad L - d \qquad \longrightarrow |$$

What answer do you find for the force on the middle plate?

6. **Casimir force is regulator-independent.** [Bonus problem] Suppose we use a different regulator for the sum in the vacuum energy $\sum_j \hbar\omega_j$. Instead we replace

$$f(d) \rightsquigarrow \frac{1}{2}\sum_{j=1}^{\infty} \omega_j K(\omega_j)$$

where the function K is

$$K(\omega) = \sum_\alpha c_\alpha \frac{\Lambda_\alpha}{\omega + \Lambda_\alpha}.$$

We impose two conditions on the parameters c_α, Λ_α:

- We want the low-frequency answer to but unmodified:

$$K(\omega) \overset{\omega \to 0}{\to} 1$$

  – this requires $\sum_\alpha c_\alpha = 1$.

- We want the sum over j to converge; this requires that $K(\omega)$ falls off faster than ω^{-2}. Taylor expanding in the limit $\omega \gg \Lambda_\alpha$, we have

$$K(\omega) \overset{\omega \to \infty}{\to} \frac{1}{\omega}\sum_\alpha c_\alpha \Lambda_\alpha - \frac{1}{\omega^2}\sum_\alpha c_\alpha \Lambda_\alpha^2 + \cdots .$$

  So we also require $\sum_\alpha c_\alpha \Lambda_\alpha = 0$ and $\sum_\alpha c_\alpha \Lambda_\alpha^2 = 0$.

First, verify the previous claims about $K(\omega)$.

Then compute $f(d)$ and show that with these assumptions, the Casimir force is independent of the parameters c_α, Λ_α.

[A hint for doing the sum: use the identity

$$\frac{1}{X} = \int_0^\infty ds e^{-sX}$$

inside the sum to make it a geometric series. To do the remaining integral over s, Taylor expand the integrand in the regime of interest.]