Characters:

for $SU(2)$ or $SO(3)$: $\chi_j(e^{i\psi J^z}) = \sum_{m=-j}^{j} e^{i\psi m}$

$= \frac{\sin(j+\frac{1}{2})\psi}{\sin\psi/2}$

class function

$$\int_{SO(3)} d\mu(g)\, \chi(g)$$

$$= \# \int_0^{\pi} d\psi\, \sin^2\psi/2\, \chi(\psi)$$

$$\int_{SU(2)} d\mu(g)\chi(g) = \# \int_0^{2\pi} d\psi\, \sin^2\psi/2\, \chi(\psi).$$

$$\langle \chi_j | \chi_{j'} \rangle = \int d\mu(g)\, \bar{\chi}_j(g)\chi_{j'}(g)$$

$$\overset{SO(3)}{=} c\int_0^{\pi} d\psi\, \cancel{\sin^2\psi/2}\, \frac{\sin(j+\frac{1}{2})\psi}{\cancel{\sin\psi/2}}\, \frac{\sin(j'+\frac{1}{2})\psi}{\cancel{\sin\psi/2}}$$

$$= c\, \frac{\pi}{2}\, \delta_{jj'} \qquad \left(\Rightarrow c = \frac{2}{\pi}\right)$$

$$R_j \otimes 2 \overset{j>0}{=} R_{j-\frac{1}{2}} \oplus R_{j+\frac{1}{2}}$$

$$\chi_j(\psi)\chi_{\frac{1}{2}}(\psi) = \sum_{m=-j}^{j} e^{i\psi m}\left(e^{-i\psi/2} + e^{i\psi/2}\right)$$

$$= \sum_{m=-j-\frac{1}{2}}^{j+\frac{1}{2}} e^{i\psi m} + \sum_{m=-j+\frac{1}{2}}^{j-\frac{1}{2}} e^{i\psi m} = \chi_{j-\frac{1}{2}} + \chi_{j+\frac{1}{2}}.$$

$$U_{s-1}(z = \cos\theta = \cos k/2) = \frac{\sin s\theta}{\sin\theta}$$ Chebyshev polynomial in z.

satisfy

$$\chi_{j_1}\chi_{j_2} = \sum_{j_3 = |-j_1+j_2|}^{j_1+j_2} \chi_{j_3}$$

Any compact Lie group:

G.O.T. (Peter-Weyl Thm):

$$\int d\mu(g)\, (D^a_{ij}(g))^* \,(D^b_{lm}(g)) = \frac{1}{d_a}\delta^{ab}\delta_{il}\delta_{jm}$$

claim: Any element $g \in G$ is conjugate to some element in $T = U(1)^r$ the Cartan subgroup.

ie $\exists k$. $g = k\, e^{i\theta\cdot H} k^{-1}$ $k \in G$.

$$\Rightarrow \chi_R(z_a \equiv e^{i\theta_a}) = \mathrm{tr}_R\, e^{i\theta\cdot H} = \sum_{\mu} e^{i\theta\cdot\mu}\, n_R(\mu)$$

$a = 1..\operatorname{rank} G$ $\qquad$ μ weights of R

eg: $G = SU(n)$. $z_a = e^{i\theta_a}$ $a = 1 \dots n$

w/ $\prod_{a=1}^{n} z_a = 1$.

wts of $\square$ are $+e_a$, $a = 1 \dots n$

$\to \chi_{\square}(z) = \sum_a z_a$

wts of $\boxed{\vdots}\}k$ are $e_{a_1} + e_{a_2} \dots + e_{a_k}$, $a_1 < a_2 < \dots < a_k$

$$\Rightarrow \chi_{\boxed{\vdots}\}k}(z) = \sum_{a_1 < a_2 \dots < a_k} z_{a_1} z_{a_2} \dots z_{a_k}$$

$= E_k(z)$ kth elementary symmetric poly'n.

$\chi_\lambda = S_\lambda(z)$ Schur polynomial.

Weyl character formula:

$$\chi_{R_\mu}(z) = \frac{A_{\mu+\rho}(z)}{\Delta(z)}$$

$$\Delta = \prod_{\alpha \in R_+} \left(e^{i\alpha\cdot\theta/2} - e^{-i\alpha\cdot\theta/2}\right)$$

$\overset{SU(2)}{=} 2i \sin \psi/2$.

$$A_\mu(z) = \sum_{w \in \mathcal{W}} (-1)^w e^{i\theta \cdot w(\mu)}$$

$$\rho = \frac{1}{2} \sum_{\alpha \in R_+} \alpha .$$

Weyl Integration formula:

$$\int_G d\mu(g)\, \underset{\substack{\uparrow \\ \text{class f'n}}}{F(g)} = \int_T d\mu(z)\, F(z) \underbrace{|\Delta(z)|^2}_{\overset{SU(2)}{=} \sin^2 \psi/2} .$$

4.3 Unification.

Isospin: $\frac{M_n - M_p}{M_n} = 0.0014$

$N = \begin{pmatrix} p \\ n \end{pmatrix}$ $\underline{2}$ of $SU(2)_I$.

$|p,\alpha\rangle = a^\dagger_{\frac{1}{2}\alpha}|0\rangle$ $\quad |n,\alpha\rangle = a^\dagger_{-\frac{1}{2}\alpha}|0\rangle$

$m_I = \pm \frac{1}{2}$

$a|0\rangle = 0.$ $\quad \{a, a^\dagger\} = \delta$, $\{a, a\} = 0$

$$\hat{T}^A |m\alpha\rangle = |m'\alpha\rangle \left(\tfrac{1}{2}\sigma^A\right)_{m'm}$$

one nucleon

$$\hat{T}^A |0\rangle = 0.$$

$$\Longrightarrow [\hat{T}^A, a^\dagger_{m\alpha}] = a^\dagger_{m'\alpha} \left(\tfrac{1}{2}\sigma^A\right)_{m'm}$$

$$\Longrightarrow \hat{T}^A = a^\dagger_{m'\alpha} \left(\frac{\sigma^a}{2}\right)_{m'm} a_{m\alpha} + \ldots$$

ind. of a.

<u>Pions</u> carry isospin

$$p \to n + \pi^+ \qquad n \to p + \pi^-$$

LHS: $I = \frac{1}{2}$ RHS: $\frac{1}{2} \otimes I_\pi \Rightarrow I_\pi = 1$

$\pi^-, \pi^0, \pi^+ \in \underline{3}$ of $SU(2)_I$

$$Q = I_3 + \tfrac{1}{2} Y \qquad Y \equiv \text{hypercharge} = 1 \text{ for nucleons}, \ 0 \text{ for pions.}$$

$|$ one π w isospin $J^3 = m = \pm 1, 0\rangle$

$$= b^\dagger_{m\alpha} |0\rangle$$

$$[b, b^\dagger] = \delta, \quad [b, b] = 0.$$

$$\hat{T}^A = \hat{T}^A_{\text{nucleons}} + b^\dagger_{m'\alpha} (J_1^A)_{m'm} b_{m\alpha} + \ldots$$

$$= \sum_{\substack{\text{particle} \\ \text{types } x}} a^\dagger_{x m' \alpha} \left(J^A_{\underline{j_x}}\right)_{m'm} a_{x m \alpha}$$

satisfy $[\hat{T}^A, \hat{T}^B] = i \underline{f^{ABC}} \hat{T}^C$.

$SU(2)_I$ is an approximate sym

$$H = H_0 + \Delta H$$

$[H_0, \hat{T}^A] = 0$

$[\Delta H, \hat{T}^A] \neq 0$

E&M, $\Delta m_{\bar{q}q}$.

$H_0 \ni QCD$

$$\frac{\sigma(p+p \to d + \pi^+)}{\sigma(p+n \to d + \pi^0)} = \frac{|\langle 1,1|1 1\rangle|^2}{|\langle 1,0|(|10\rangle - |00\rangle)/\sqrt{2}|^2} = 2.$$

$|p+p\rangle = |\tfrac{1}{2}\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2}\tfrac{1}{2}\rangle = |1,1\rangle$

$|p,n\rangle = |\tfrac{1}{2}\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2} -\tfrac{1}{2}\rangle = \underline{\tfrac{1}{\sqrt{2}}}(|1,0\rangle - |0,0\rangle)$

<u>π-N Coupling</u>: $\pi = \vec{\pi}\cdot\vec{\sigma} = \begin{pmatrix} \pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & -\pi^0 \end{pmatrix}$

$$\xrightarrow{SU(2)_I} U \pi U^\dagger$$

$\Rightarrow \mathcal{L} = \dots g \bar{N}_i \pi^i{}_j N^j$ is $SU(2)_I$ inv't

$$= g\left(\bar{p}\pi^0 p - \bar{n}\pi^0 n + \sqrt{2}\left(\bar{p}\pi^+ n + \bar{n}\pi^- p\right)\right)$$

iso spin $\frac{3}{2}$

$$\frac{\sigma(\pi^+ + p \to \Delta^{++} \to \pi^+ + p)}{\sigma(\pi^- + p \to \Delta^0 \to \pi^- + p)} = 3$$

isospin $\frac{3}{2}$ mult $\ni (\Delta^{++}, \Delta^+, \Delta^0, \Delta^-)$

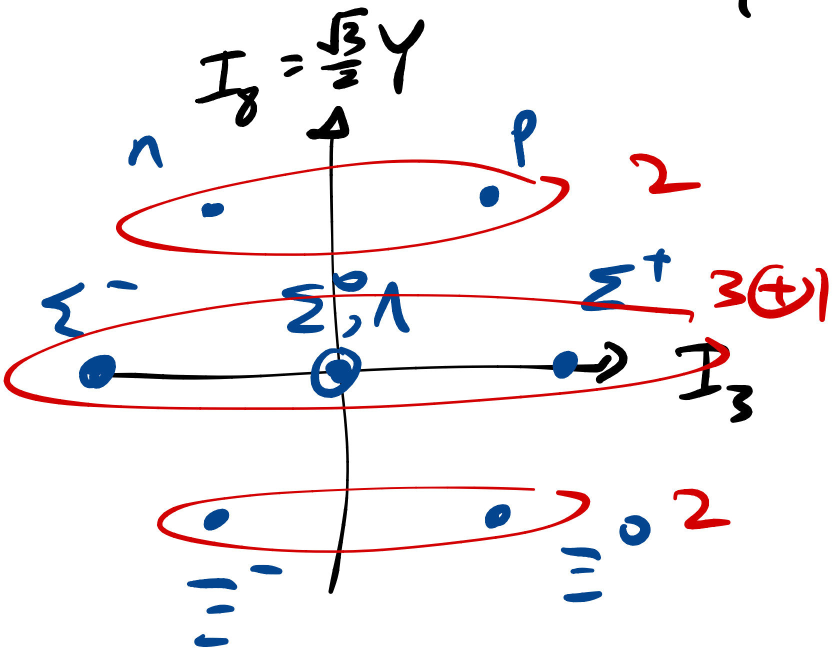
<u>Eight-fold way</u>. Another baryon Λ $M_\Lambda \sim 1.115$ GeV

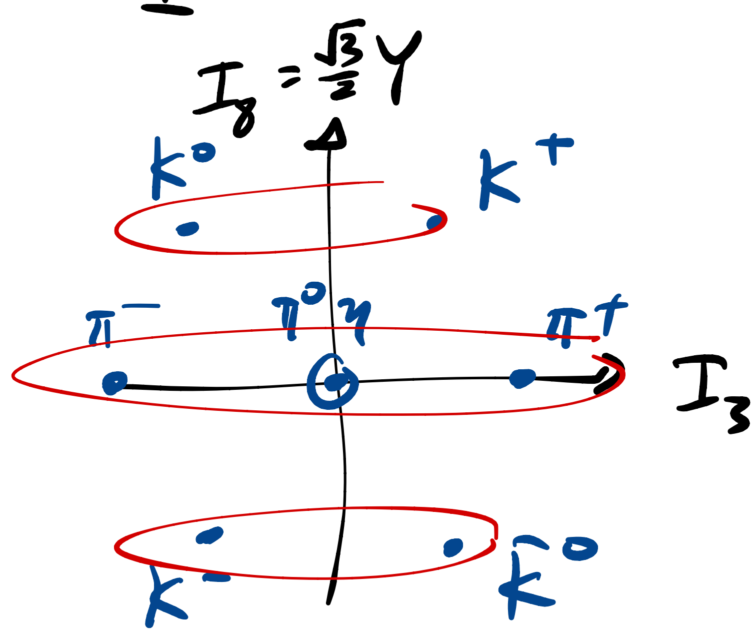
$\begin{pmatrix} n \\ p \\ \Lambda \end{pmatrix}$?

$$\left\{ \begin{array}{ll} \Xi^\pm & 2 \\ & \oplus \\ \Sigma^{\pm 0} & 3 \\ & \oplus \\ \Lambda & 1 \\ & \oplus \\ N & 2 \end{array} \right.$$

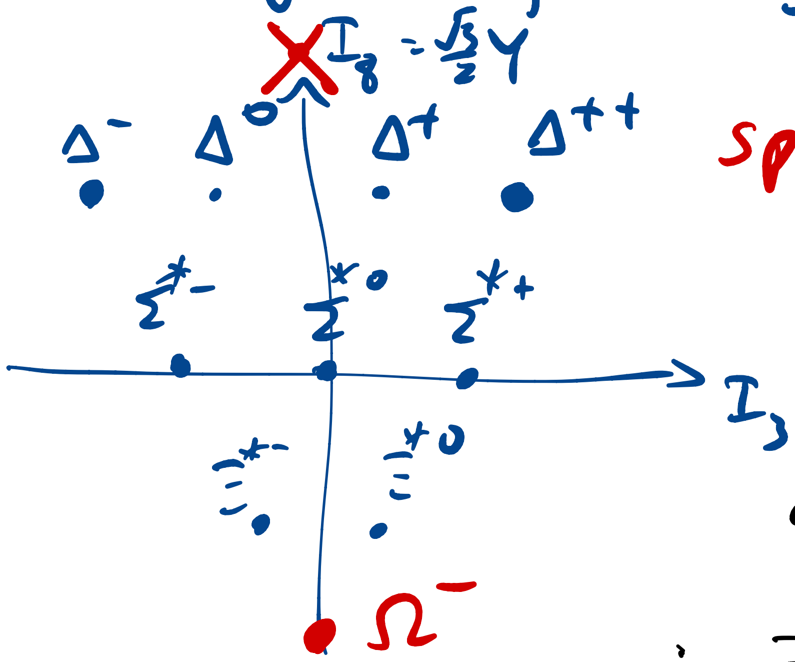
more mesons:

$\pi^{\pm 0}$	$\underline{3}$
$K^+ K^0$	$\underline{2}$
$\bar{K}^0 K^-$	$\underline{2}$
η	$\underline{1}$





$\underline{8}$ of $SU(3)_f \supset SU(2)_I \times U(1)_Y$ $\qquad$ $\underline{8}$ of $SU(3)_f$



spin $\frac{3}{2}$ baryon resonances.

$\underline{10} = \boxed{\;\;|\;\;|\;\;} = R_{3\mu'}$

$\dim(\boxed{3|4|5}) = \frac{3 \cdot 4 \cdot 5}{3 \cdot 2} = 10$ ✓.

Gell-Mann Okubo mass formula.

guess : $\Delta H_{SU(3)_f \to SU(2)_I} \propto \underline{\underline{I_8}}$

Spin $\frac{1}{2}$ baryons

$$B \equiv \lambda^A \frac{B^A}{\sqrt{2}} = \begin{pmatrix} \Sigma^0/\sqrt{2} + \Lambda^0/\sqrt{6} & \Sigma^+ & p \\ \Sigma^- & -\Sigma^0/\sqrt{2} + \Lambda^0/\sqrt{6} & n \\ \Xi^- & \Xi^0 & -2\Lambda^0/\sqrt{6} \end{pmatrix}$$

mass term : $M_0 \,\mathrm{tr}\, B^\dagger B$ SU(3) inv't.

$$\underline{\underline{m_1}} \,\mathrm{tr}\, B^\dagger I_8 B + \underline{\underline{m_2}} \,\mathrm{tr}\, B^\dagger B I_8$$

$$\underline{8} = \underline{2} \oplus \underline{3} \oplus \underline{1} \oplus 2 \quad \underline{\underline{4}} \text{ isospin multiplets.}$$

Spin $\frac{3}{2}$ baryons: $\underline{10}$ of SU(3). Do pert. thy.

$$\Delta E \sim \underline{\langle 10 | I_8 | 10 \rangle}$$

$$8 \otimes 10 = 8 \oplus 10 \oplus 27 \oplus 35$$

$\Rightarrow$ only $\underline{\text{one}}$ mass term $\propto$ hypercharge.

$$M_\Sigma - M_\Delta = M_\Xi - M_\Sigma = M_\Omega - M_\Xi .$$

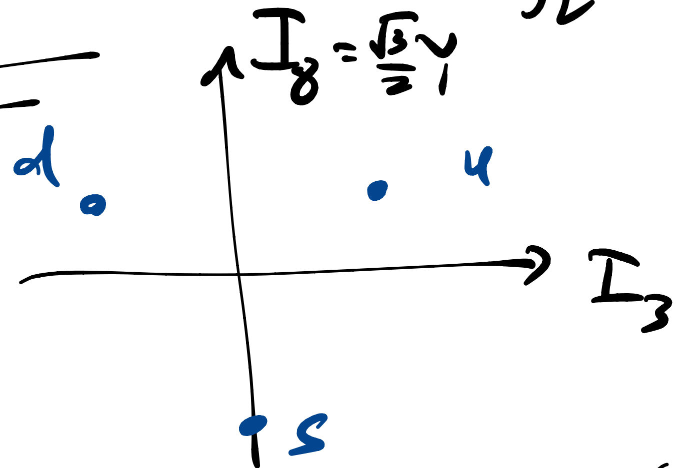
$M_\Delta \sim 1230$ MeV

$M_\Sigma \sim 1385$

$M_\Xi = 1530$ $\longrightarrow$ $M_\Omega \simeq 1680$

$M_\Omega^{expt} \simeq 1672$ MeV.

<u>Quarks</u> :



$\in \underline{3}$ of $SU(3)_f$

$\underline{B_{abc}} \sim \underline{q_a q_b q_c}$

$\square \otimes \square \otimes \square = (\ydiagram{1,1} \oplus \ydiagram{2}) \otimes \square$

$3 \otimes 3 \otimes 3$

$= \ydiagram{1,1,1} \oplus \ydiagram{2,1} \oplus \ydiagram{2,1} \oplus \ydiagram{3}$ ← Symmetric

$\quad 1 \quad\ \ 8 \quad\ \ 8 \quad\ \ 10$

spin $\frac{1}{2}$ baryons (↑ 8)   spin $\frac{3}{2}$ baryons (↑ 10)

$M_{ab} \sim q_a \bar{q}_b$

$\square \otimes \ydiagram{1,1} = \bullet \oplus \ydiagram{2,1}$

$3 \otimes \bar{3} = 1 \oplus \underline{8}$ ← mesons

<u>problem</u> : ① B's are fermions $\Rightarrow$ q's are fermions.

② where are the quarks?! (they have <u>fractional Q.</u>)

Color. q's are also a $\underline{3}$ of $SU(3)_c$

$q_{a\,i}$

$a = 1..3$ (u, d, s) flavor; $i = 1..3$ (r, g, b) color

* law: finite energy particles are colorless.

$B = \epsilon_{ijk}\, q^i q^j q^k$ is a color singlet in $3_c \otimes 3_c \otimes 3_c$

$M = q^i \bar{q}_i$ is a color singlet in $3_c \otimes \bar{3}_c$

* why?

In e&m $H \propto q_1 q_2 \rightsquigarrow \sum_A T_A^{R_1} \otimes T_A^{R_2}$

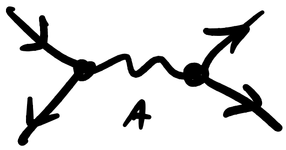
gauge theory w/ $G = U(1)$ — charge is a # δ

$\Downarrow$ w/ G — $\Downarrow$ $(T^A)_i{}^j$ representation matrix

$$\hat{T}_A |i\rangle_1 = |j\rangle (T_A^{R_1})^i{}_j \qquad \hat{T}_A |x\rangle_2 = |y\rangle_2 (T_A^{R_2})^x{}_y$$

2-particle state $\quad v_{xi}\, |i\rangle_1 \otimes |x\rangle_2$

$$H_{U(1)} = q_1 q_2 \rightsquigarrow H_{QCD} = \sum_A T_A^{R_1} \otimes T_A^{R_2}$$

$=$  $\equiv H$

$$\hat{T}_A = T_A^{R_1} + T_A^{R_2} \quad \underline{\text{claim:}}\ [H, \hat{T}_A] = 0.$$

$$C_2^R = \sum_A T_A^R T_A^R \equiv T_R^2$$

$$H = \frac{1}{2}\Big(\underbrace{T^2}_{\substack{\text{depends}\\ \text{on the}\\ \text{combined state}}} - T_1^2 - T_2^2\Big)$$

T_1^2, T_2^2: completely specified by 1, 2.

minimized by smaller Reps. of the combined system.

$1 = q \quad 2 = \bar{q} \longrightarrow$ minimized by singlet

$1 = q \quad 2 = q \longrightarrow$ min. by $\bar{3} \Longrightarrow$

$\epsilon_{ijk}\, q^i q^j q^k$ minimizes $H_{12} + H_{23} + H_{31}$.

4.3.3 Grand Unification

$$G_{SM} = SU(3)_c \times SU(2) \times U(1)$$

weak interactions. $\propto$ hypercharge

label a rep:

$$(D, d)_{R, s}$$ charge under $U(1)$

rep of $SU(3)$ rep of $SU(2)$

$$\begin{cases} [\hat{T}_A, a^\dagger_{x,i}] = a^\dagger_{y,i} (T^D_A)_{yx} \\ [\hat{R}_A, a^\dagger_{x,i}] = a^\dagger_{x,j} (R^d_A)_{ji} \\ [S, a^\dagger_{x,i}] = s\, a^\dagger_{x,i} . \end{cases}$$

electric charge is

$$Q = R_3 + S$$
$$= R_3 + Y/2$$

A generation of the SM:

$$R_{gen} = \underset{u_R}{(3,1)_{2/3}} \oplus \underset{d_R}{(3,1)_{-1/3}} \oplus \underset{e_R}{(1,1)_{-1}} \oplus \underset{q_L^c}{(\bar{3},2)_{-1/6}} \oplus \underset{e_L^c = (e_L, \nu_L)^c}{(1,2)_{1/2}}$$

$H \in \underline{2}$ of $SU(2)$, maybe a heavy $\nu_R \in (1,1)_0$.

Q: $\exists\, G \supset G_{SM}$ s.t. R_{gen} is an irrep?
compact, simple...

$$\operatorname{tr}_{R_{gen}} S = \tfrac{2}{3}\cdot 3 + -\tfrac{1}{3}\cdot 3 - 1 + 6\cdot\left(-\tfrac{1}{6}\right) + \tfrac{1}{2}\cdot 2$$
$$= 2 - 1 - 1 - 1 + 1 = 0.$$

$\operatorname{rank} G \geq 4.$ Cartan of G_{SM}
$= \{ T_3, T_8, R_3, S \}$

$SU(5) \supset SU(3) \times SU(2) \times U(1)$

$$\begin{pmatrix} 5 \end{pmatrix} = \begin{pmatrix} 3 \\ \hline 2 \end{pmatrix}$$

$$\begin{array}{cc} & \begin{array}{cc} 3 & 2 \end{array} \\ \begin{array}{c} 3 \\ 2 \end{array} & \left(\begin{array}{c|c} T_A & 0 \\ \hline 0 & 0 \end{array}\right) \end{array} \quad \begin{array}{c} \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & R_A \end{array}\right) \\ \begin{array}{cc} 3 & 2 \end{array} \end{array} \quad \left(\begin{array}{c|c} -\mathbb{1}_3/3 & \\ \hline & +\mathbb{1}_2/2 \end{array}\right) = S$$

$$\underline{5} = (3,1)_{-1/3} \oplus (1,2)_{+1/2} = d_R \oplus e_L^c$$

$$R_{\tiny\begin{array}{|c|}\hline \\ \hline \\ \hline\end{array}} = \Lambda^2 5 = \underline{10} \qquad R_{\tiny\begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline\end{array}} = \Lambda^3 5 = \Lambda^2 \bar{5} = \underline{\overline{10}}.$$

$$\Lambda^2 5 = \left(\underline{(3,1)_{-1/3}} \oplus (1,2)_{1/2} \right) \otimes_{AS} \left((3,1)_{-1/3} \oplus \underline{(1,2)_{1/2}} \right)$$

$$= (\underbrace{\Lambda^2 3}_{=\bar{3}}, 1)_{-2/3} \oplus (1, \underbrace{\Lambda^2 2}_{=\underline{1}})_1$$

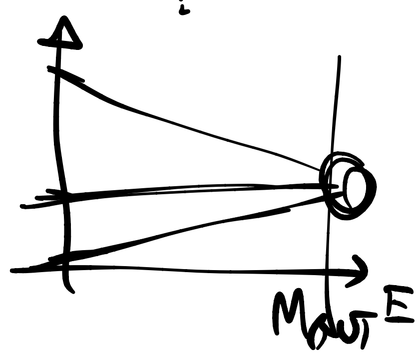
$$\oplus\ (3,2)_{-\frac{1}{3}+\frac{1}{2}=\frac{1}{6}}$$

$$= (\bar{3},1)_{-2/3} \oplus (1,1)_1 \oplus (3,2)_{1/6}$$

$$= \left((3,1)_{2/3} \oplus (1,1)_{-1} \oplus (\bar{3},2)_{-1/6} \right)^*$$

$$= \left(u_R \oplus e_R \oplus q_L^c \right)^*$$

$$\Rightarrow \boxed{R_{gen} = \overline{10} \oplus 5} \quad !$$

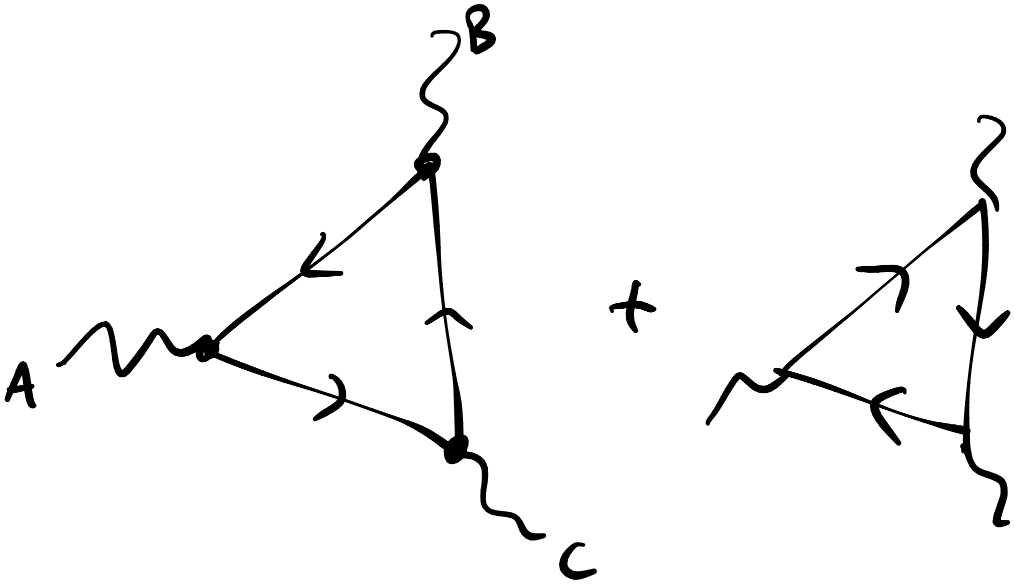


$SU(5) \subset SO(10)$

$$16 = \square \oplus \begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline \end{array} \oplus \begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline \\ \hline \\ \hline \end{array} = 5 \oplus \overline{10} \oplus 1$$

$= R_{gen} \oplus$

Right-handed neutrino.



$\overset{!}{=} 0$

$$= d_{ABC}^{(R)} \text{ symmetric tensor.}$$

"gauge anomaly must cancel".

(In SM, also global anomalies cancel.)