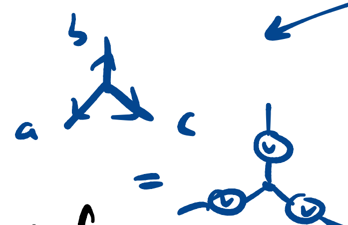
Birdtracks, cont'd: An invariant tensor of G (symbol)

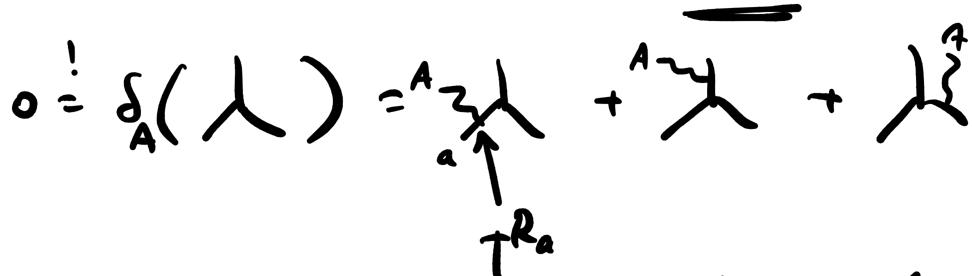
is d s.t. $\overset{\text{e.g.}}{\forall}$ $d^{abc} v_a w_b x_c \overset{!}{=} d^{abc} v'_a w'_b x'_c$



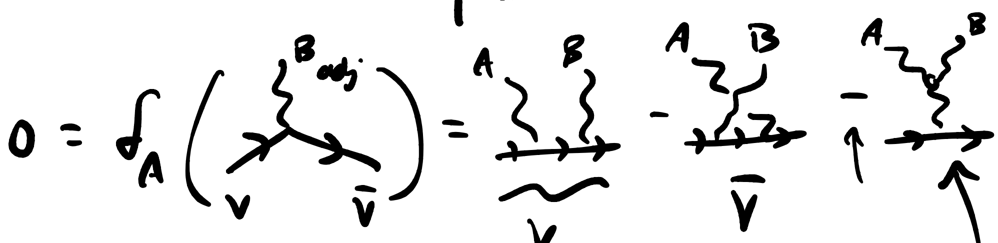
w/ $v'_a = (U^{R_v})_a{}^{a'} v_{a'}$

$\forall$ U representing G.

Infinitesimal $U = e^{i\theta^A T^A} = 1 + i\underline{\theta^A T^A} + \dots$



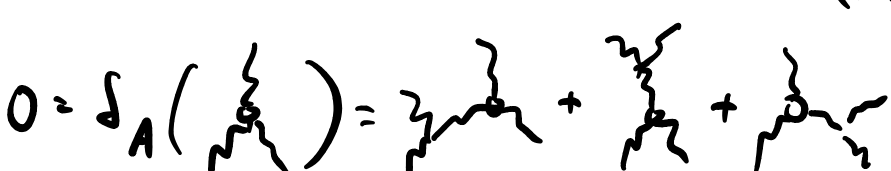
eg:



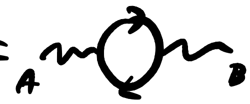
has generator $-(T^V)^*$

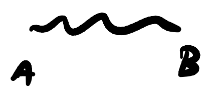
has gen. $-i f_{ABC}$

i.e. $[T^A, T^B] = i f^{ABC} T^C$.



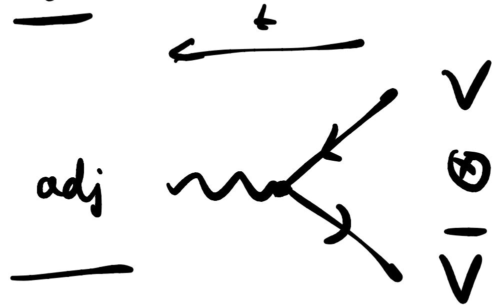
= Jacobi id.

SU(n): $\operatorname{tr} T^A T^B = (T^A)^j{}_k (T^B)^k{}_j =$ 

$= T_R$ 

$\operatorname{tr}\left[(\text{Lie alg})\, T^C\right] \Rightarrow$

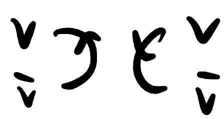
Q: how could we have discovered T_A?



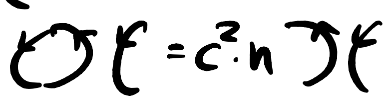
$\Rightarrow V \otimes \bar{V} = \text{adj} \oplus \cdots$ (irreps)

$\Rightarrow \mathrm{id}_{V\otimes\bar{V}} = \underline{P_{adj}} \oplus P_{others}$

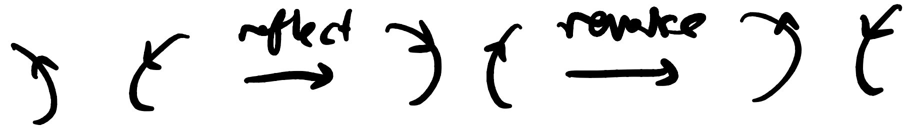
= eigenspaces of <u>any</u> hermitian invt tensor on $V \otimes \bar{V}$

eg: 

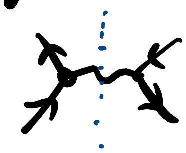
$P_1 = c$ 

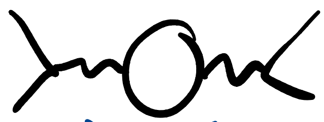
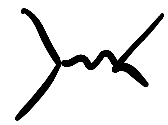
$P_1^2 = c^2$  $= c^2 \cdot n$ 

$(\text{diagram})^\dagger = \left(\begin{array}{l}\text{reflect in y axis}\\ \text{reverse arrows}\end{array}\right)$



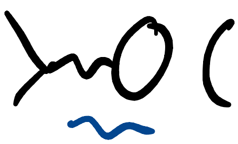
In fact: $P_{adj} = \mathbb{1}_{V\otimes\bar{V}} - P_1$

$P_{adj} = c$ 

$P_{adj}^2 = c^2$  $= c^2 T_R$ 

$= T_R$ 

$\Rightarrow c = 1/T_R$.

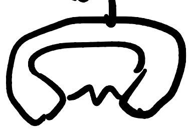
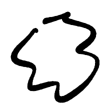
$P_{adj} P_1 =$  $= 0$

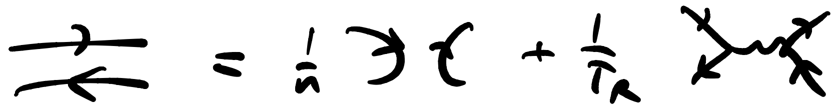
 $= \operatorname{tr} T = 0$.

$\dim R_1 = \operatorname{tr} P_1 = \operatorname{tr}$  $/n$ $=$  $/n = n/n = 1$.

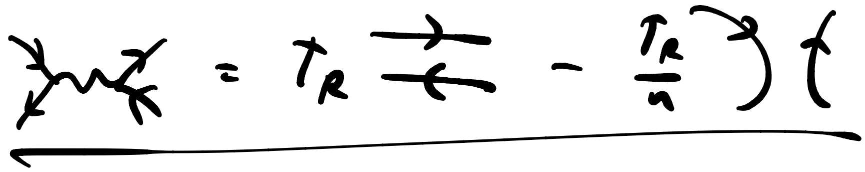
singlet.

$\dim R_{adj} = \operatorname{tr} P_{adj}$

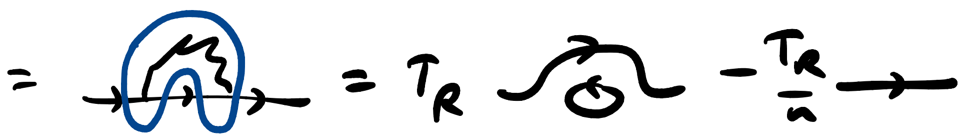
$= \frac{1}{T_R}$  $= \frac{1}{T_R}$  $= T_R$  $=$  $= n^2 - 1$.



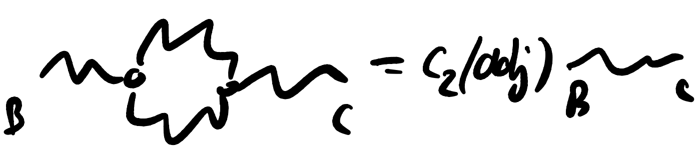
$\mathrm{id}_{n\otimes\bar{n}}$

i.e. 

eg: $(T^A_\square T^A_\square)^j{}_\ell \overset{\text{Schur}}{=} C_2(\square)\,\delta^j{}_\ell$

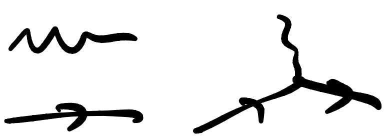


$$= \underbrace{T_R \frac{n^2-1}{n}}_{=C_2(\square)} \longrightarrow$$

$(T^A_{adj} T^A_{adj})^B{}_C =$ 

$$C_2(adj) = n\,2T_R .$$

Feynman diagrams for gauge thy:



BIRDTRACKS = color factors.

Identical particles: irreps of S_k $\leftrightarrow$ irreps of G

one particle $|\psi\rangle \in \text{span}\{|i\rangle \; i=1..n\} = \square$

$$g : |\psi\rangle \to D(g)|\psi\rangle$$

k particles: If distinguishable

$$\square^{\otimes k} \ni |\psi_1\rangle \otimes \ldots \otimes |\psi_k\rangle \mapsto D(g)|\psi_1\rangle \otimes \ldots \otimes D(g)|\psi_k\rangle$$
$$= D^{(k)}(g)|\psi_1\rangle \otimes \ldots \otimes |\psi_k\rangle$$

$\exists$ indistinguishable particles $\to$ (Anti) symmetrization produces invariant subspaces.

$\square^{\otimes k}$ is also a rep of S_k

$$R(\pi)\, |\psi_1\rangle \otimes \ldots \otimes |\psi_k\rangle = |\psi_{\pi_1}\rangle \otimes \ldots \otimes |\psi_{\pi_k}\rangle$$

and $[R(\pi), D^{(k)}(g)] = 0 \quad \forall g, \pi.$

eg: $k=2$, $n=2$. $\quad 2 \otimes 2 = 1 \oplus 3$

$$P_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}(V_2 \otimes V_2) = V_1 \quad P_{\square\square}(V_2 \otimes V_2) = V_3.$$

$$\dim R^{SU(n)}_{\lambda = \text{young diagram}} \neq \dim R^{S_k}_{\lambda}$$

$= \#$ of ways of placing $\{1 \dots n\}$ in the diagram

preserving order in rows

strict order in cols

$$\begin{array}{|c|}\hline 3\\\hline 2\\\hline\end{array}\quad \begin{array}{|c|}\hline 3\\\hline 1\\\hline\end{array}\quad \begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array} \longrightarrow \bar{3}$$

$$\boxed{3}\quad \boxed{2}\quad \boxed{1} \longrightarrow 3$$

$$\begin{array}{|c|c|}\hline 3&3\\\hline\end{array}\ \begin{array}{|c|c|}\hline 3&2\\\hline\end{array}\ \begin{array}{|c|c|}\hline 3&1\\\hline\end{array}\ \begin{array}{|c|c|}\hline 2&2\\\hline\end{array}\ \begin{array}{|c|c|}\hline 2&1\\\hline\end{array}\ \begin{array}{|c|c|}\hline 1&1\\\hline\end{array} \longrightarrow 6$$

$\vdots$

$= \#$ tableaux made from λ

$$\left\{ \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array} \right.$$

$k = \#$ of boxes

$h_\lambda = \prod$ hooks

$$= \frac{k!}{h_\lambda}$$

$$\begin{array}{|c|c|}\hline 3&1\\\hline 1\\\cline{1-1}\end{array} \quad \frac{6}{3} = 2$$

<u>factors over hooks rule</u> : $\dim R^{SU(n)}_{\lambda} = \dfrac{f_\lambda}{h_\lambda}$

$$\begin{array}{|c|c|c|}\hline n&n+1&n+2\\\hline n-1&n\\\cline{1-2}\end{array} \qquad f_\lambda = \prod(\#\text{s in the boxes})$$

$$SU(n):\ \begin{array}{|c|}\hline n\\\hline n-1\\\hline\end{array} \rightarrow \frac{n(n-1)}{2} = \dim \Lambda^2 \underline{n}$$

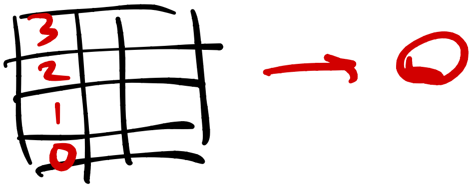
$$\begin{array}{|c|c|}\hline n&n+1\\\hline\end{array} \rightarrow \frac{n(n+1)}{2} = \dim \mathrm{Sym}^2 n.$$

in $SU(2)$

$$\underbrace{\begin{array}{|c|c|c|}\hline 2&3&4\\\hline 1&2\\\cline{1-2}\end{array}}\rightarrow \frac{2\cdot\cancel{3}\cdot\cancel{4}\cdot\cancel{2}}{\cancel{4}\cdot\cancel{3}\cdot\cancel{2}} = 2\ \checkmark \qquad \underset{\text{boxes}}{n}\left\{\begin{array}{|c|}\hline \\\hline \\\hline \\\hline\end{array}\right. = \Lambda^n n = 1.$$

$$\begin{array}{|c|c|c|}\hline 4&3&1\\\hline 2&1\\\cline{1-2}\end{array}$$

for SU(3)



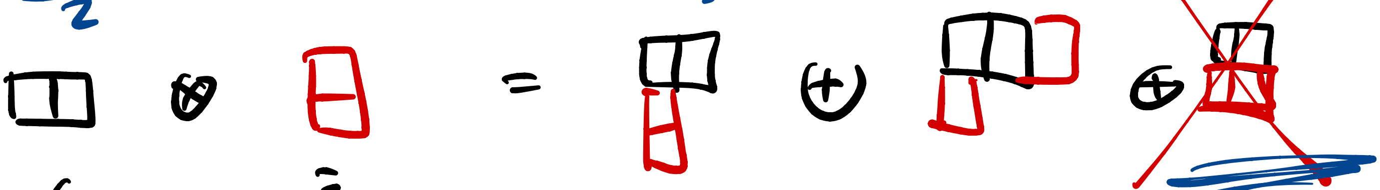
Multiplying irreps.

$R_\lambda \otimes R_{\lambda'} = \bigoplus_\mu R_\mu$

for SU(3)

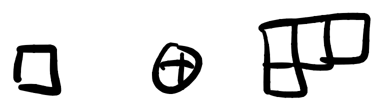
$6 \oplus 3 = 8 \oplus 10$

for SU(n): $\frac{n(n+1)}{2} + n = \frac{n(n-1)(n+1)}{3} + \frac{n(n+1)(n+2)}{6}$ ✓

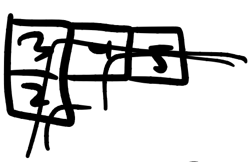


6 3

in SU(3):  $=1$

$=$ 

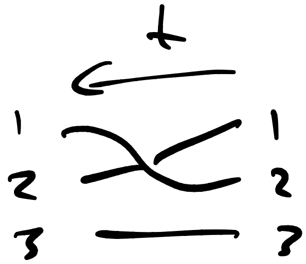
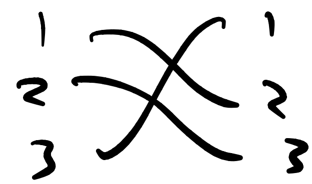
3 15 ✓

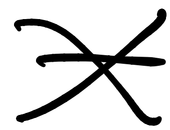


$$\frac{3\cdot 4\cdot 5\cdot 2}{4\cdot 2} = 15$$

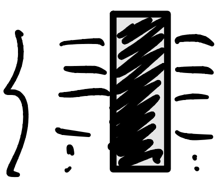
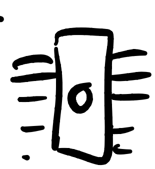
Schur-Weyl duality

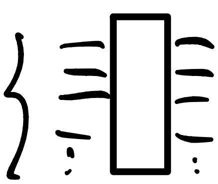
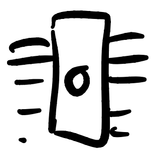
birdtracks depict the group algebra $\mathbb{C}[S_k]$ of S_k (= brauer algebra)

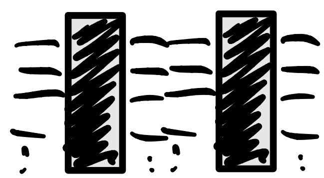
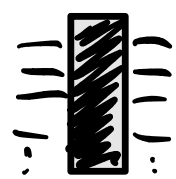
eg: $(12) \in S_3$  (123) : 

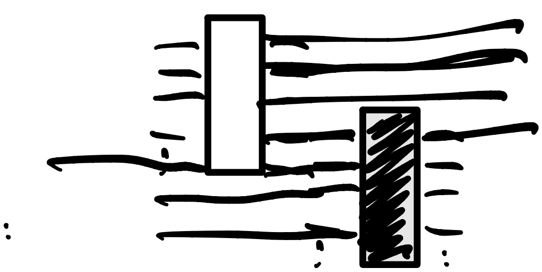
$(12)(123) =$  $=$  $= (13)$.

projections:

k lines  $= \frac{1}{k!} \sum_{\sigma \in S_n} (-1)^{\sigma}$ 

k lines  $= \frac{1}{k!} \sum_{\sigma \in S_n}$ 

 $=$ 

 $= 0$.

$\lambda \vdash k$ Young diagrams w/ k boxes $\longleftrightarrow$ irreps of S_k

$$R_\lambda = \mathrm{span}\{|\text{tableaux on } \lambda, \hat{\lambda}\rangle\}$$

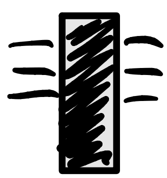
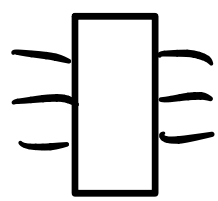
$$R_{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}} = \mathrm{span}\left\{\left|\begin{smallmatrix}1&2\\3\end{smallmatrix}\right\rangle, \left|\begin{smallmatrix}1&3\\2\end{smallmatrix}\right\rangle\right\}$$

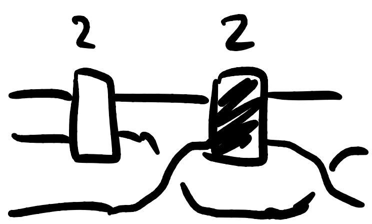
each tableau $\hat{\lambda}$

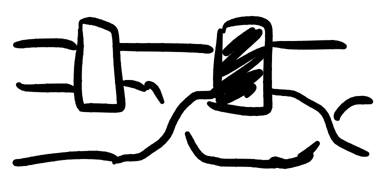
$$\longrightarrow Y_{\hat{\lambda}} \in \mathbb{C}[S_k]$$

Young symmetrizers

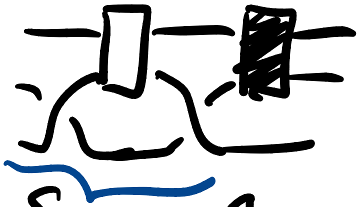
$$Y_{\hat{\lambda}} = \frac{1}{h_\lambda} s_{\hat{\lambda}} a_{\hat{\lambda}}$$

$Y_{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}} =$  $Y_{\begin{smallmatrix}\square\square\square\end{smallmatrix}} =$ 

$Y_{\begin{smallmatrix}1&2\\3\end{smallmatrix}} = \frac{1}{3}$ 

$= \frac{4}{3}$ 

s_{12} a_{13}

$Y_{\begin{smallmatrix}1&3\\2&\end{smallmatrix}} = \frac{4}{3}$ 

S_{13} a_{12}

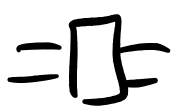
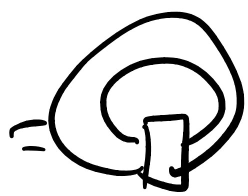
claim: either one projects $\square^{\otimes 3}$ to $R_{\begin{smallmatrix}\square&\square\\\square&\end{smallmatrix}}$ (= adj of $SU(3)$)

$Y_{\hat\lambda} : n^{\otimes k} \to n^{\otimes k}$

claim: ① $\mathrm{Im}(Y_{\hat\lambda}) = R_\lambda^{SU(n)}$. ② $Y_{\hat\lambda} Y_{\hat\lambda} = Y_{\hat\lambda}$.

③ $Y_{\hat\lambda} Y_{\hat\mu} \propto \delta_{\lambda\mu}$

eg: $\mathrm{tr}\, Y_{\hat\lambda} = \dim R_\lambda^{SU(n)}$

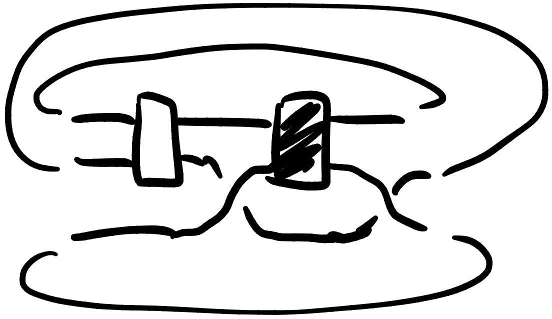
eg: $\mathrm{tr}\, Y_{\begin{smallmatrix}1&2\end{smallmatrix}} = \mathrm{tr}$  $=$ 

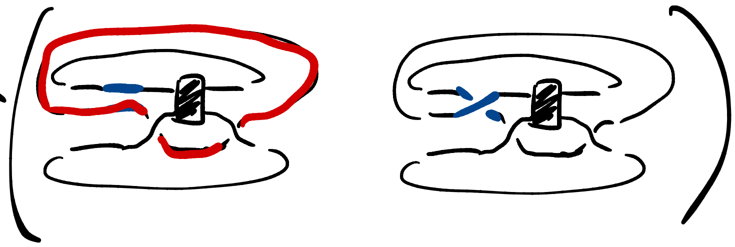
$= \frac{1}{2}\Big($  $\pm$  $\Big)$

$\mathrm{tr}\, Y_{\begin{smallmatrix}1\\2\end{smallmatrix}} = \mathrm{tr}$ 

$= \frac{1}{2}(n^2 \pm n)$

$= \frac{n(n \pm 1)}{2}$

$$\mathrm{tr}\, Y_{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}} = \frac{4}{3}$$ 

$$= \frac{2}{3}$$ 

$$= \frac{2}{3}\Big({\color{red} n}\; \mathrm{tr}\, Y_{\begin{array}{|c|}\hline \\ \hline \\ \hline\end{array}} + \underbrace{\mathrm{tr}\, Y_{\begin{array}{|c|}\hline \\ \hline \\ \hline\end{array}}}_{\frac{n(n-1)}{2}} \Big)$$

$$= \frac{n(n^2-1)}{3} = \dim R^{SU(n)}_{\begin{array}{|c|c|}\hline & \\ \hline \\ \cline{1-1}\end{array}} \quad \checkmark$$

$$\boxed{n^{\otimes k} = \bigoplus_{\lambda \vdash k} R_\lambda^{SU(n)} \otimes R_\lambda^{S_k}}$$

Schur-Weyl duality.

$$n^k = \sum_{\lambda\vdash k} \dim R_\lambda^{SU(n)} \dim R_\lambda^{S_k}.$$

ex:

$$\mathrm{tr}\, Y_{n-1\left\{\begin{array}{|c|c|}\hline & \\ \hline \\ \cline{1-1} \vdots \\ \cline{1-1} \\ \cline{1-1}\end{array}\right.} = n^2-1 = \dim(\text{adj of } SU(n)).$$

4.2 Group integration & characters

$$\frac{1}{|G|}\sum_g \cdots \rightsquigarrow \int_G d\mu(g) \cdots$$

SO(3) or SU(2). $\chi_j(g) = \mathrm{tr}_{V_j} D(g) = \chi_j(hgh^{-1})$

$$R' R(\theta,\hat{n}) (R')^T = R(\theta, \hat{n}')$$

$\Rightarrow$ can evaluate on the Cartan subgroup

$$D(\psi) = e^{i\psi J_z}$$

$$\chi_j(\psi) = \sum_{m=-j}^{j} \langle jm | e^{i\psi J_z} | jm\rangle$$

$$= \underline{\sum_{m=-j}^{j} e^{i\psi m}} = \frac{\sin(j+\frac{1}{2})\psi}{\sin \psi/2}$$

$$\chi_j(\mathbb{1}) = 2j+1 = \dim V_j \qquad \chi_0(\psi) = 1.$$

$$\langle \chi_1 | \chi_2 \rangle = \frac{1}{|G|}\sum_g \bar{\chi}_1(g)\chi_2(g) \rightsquigarrow \int_G d\mu(g)\, \bar{\chi}_1(g)\chi_2(g)$$

Crucial step: $\Lambda_X \equiv \int d\mu(g)\, D^\dagger(g)\, X\, D(g)$

has $D^\dagger(h)\,\Lambda_X\, D(h) = \Lambda_X$ is an intertwiner

$$= \int d\mu(g)\, D^\dagger(gh)\, X\, D(gh)$$
$$= \int d\mu(kh^{-1})\; D^\dagger(k)\, X\, D(k) \overset{!}{=} \Lambda_X$$

requires: $d\mu(kh^{-1}) \overset{!}{=} d\mu(k)$

G inv't measure = Haar measure.

eg: $SO(2) = U(1) = \{e^{i\theta}\}$ $\quad d\mu(e^{i\theta}) = d\theta$

SU(2) or SO(3): $U = w\mathbb{1} + i\vec{x}\cdot\vec{\sigma} \in SU(2)$

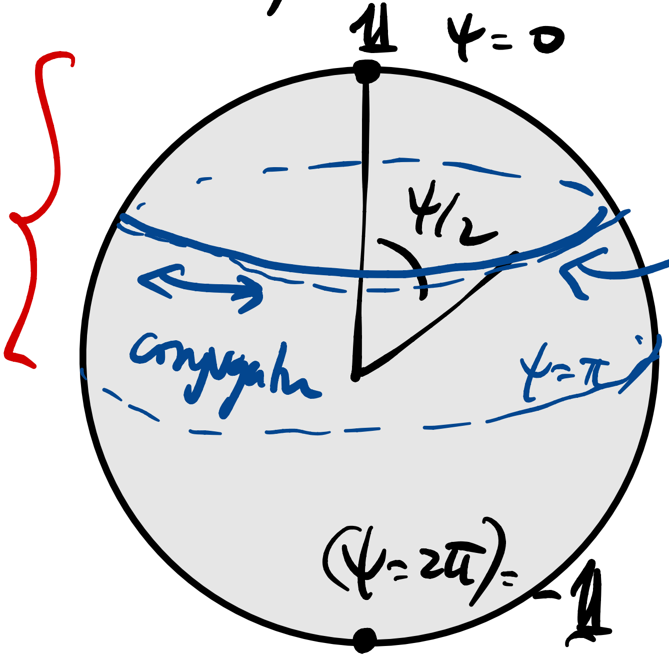
$$\iff 1 = w^2 + \vec{x}^2$$

$$U = e^{i\vec{\psi}\cdot\frac{\vec{\sigma}}{2}} = \cos\tfrac{\psi}{2} + i\hat{\psi}\cdot\vec{\sigma}\sin\tfrac{\psi}{2}$$

$$\iff w = \cos\tfrac{\psi}{2} \quad \vec{x} = \sin\tfrac{\psi}{2}\,\hat{\psi}.$$

SU(2) has $SU(2)_L \times SU(2)_R$ symmetry $= SO(4)$ $\quad h \to g_L h g_R^{-1}$
(g_L, g_R)

$$\Rightarrow d\mu(\psi,\theta,\varphi) = \underline{\sin^2\frac{\psi}{2}\, d\psi}\ \cancel{\sin\theta\, d\theta}\, d\varphi$$



$$S^2 = \{1 - w^2 = \tilde{x}^2\}$$
$$= \sin^2\frac{\psi}{2}$$

$$1 = w^2 + \tilde{x}^2$$

<u>Range of ψ?</u>

$$\psi \in [0, 2\pi)$$
$$w \in (-1, 1)$$

$$X = \tilde{x}\cdot\vec{\sigma} \qquad U_\psi X U_\psi^\dagger = \vec{x}'\cdot\vec{\sigma}$$

$$\underline{x' = R_{2\psi}x}$$

$$U, -U \longrightarrow R.$$

$$SO(3) = SU(2)/\mathbb{Z}_2 = \langle \text{antipodal map } U \to -U \rangle$$

in $SO(3)$ $\psi \in [0, \pi)$.

$$* \int_G d\mu(g)\, f(g) = \frac{1}{|W|} \int_{T = U(1)^r} d\mu(z)\, f(z) \underbrace{|\Delta(z)|^2}$$

class f'n

$$z_a = e^{i\theta_a} \qquad \Delta(z) = \prod_{\alpha \in R^+} \left(e^{i\theta\cdot\alpha} - h.c.\right)$$

$$\int_T d\mu'(z) \ldots \equiv \int_T d\mu(z)\, |\Delta(z)|^2 \ldots$$

<u>eg</u>: $SU(n)$: $\Delta = \underbrace{\prod_{i<j} (z_i - z_j)}$

$$\pi : (G/T, T) \to G$$

Weyl integration formula.

$$\Delta = \mathrm{Jac}(\pi).$$