$$\mathcal{H} = \text{span}\left\{ |s_1 \ldots s_n\rangle = |s_1\rangle \otimes |s_2\rangle \cdots \otimes |s_n\rangle \right\}$$

$$= \bigotimes_{a=1}^{n} \mathcal{H}_a$$

a qbit

generators of $SO(2n)$:

$$\underbrace{H_a = c_a^\dagger c_a - 1/2 \qquad E_{ab} = c_a^\dagger c_b}_{SU(n)} \quad , E'_{ab} = c_a^\dagger c_b^\dagger .$$

As matrices: $\quad H_a = Z_a/2 = \mathbb{1} \otimes \cdots \otimes Z_a/2 \otimes \cdots \otimes \mathbb{1}$

a-th entry

$$E_a \equiv \underline{T_{2a-1,2n+1}} - i\, \underline{T_{2a,2n+1}}$$

$$= i \left(\frac{\gamma_{2a-1} - i\gamma_{2a}}{2} \right) \gamma_F = i c_a^\dagger \gamma_F .$$

like σ_a^+ ex<u>cept</u> $\quad \{E_a, E_b\} = 0$ for $a \neq b$.

solution: Jordan-Wigner

$$E_1 = \sigma_1^+$$

$$E_2 = Z_1 \sigma_2^+$$

$$E_3 = Z_1 Z_2 \sigma_3^+$$

$$E_4 = Z_1 Z_2 Z_3 \sigma_4^+$$

$$\sigma^+ \propto X + iY$$

$$\vdots$$

$$E_a = \prod_{b=1}^{a-1} \sigma_a^+ \qquad a = 1..n.$$

$$T_{2a-1,2n} = \frac{Z_1 \cdots Z_{a-1}}{2} X_a \qquad T_{2a,2n} = \frac{Z_1 \cdots Z_{a-1}}{2} Y_a$$

general element of $SO(2n+1)$:

$$T_{ij} = -i \left[T_{i,2n+1}, T_{j,2n+1} \right]$$

$$(i, j \neq 2n+1)$$

Q: for which N is the spinor of $SO(N)$ real?

R is <u>not complex</u> if $\exists\, S \in R^{\otimes 2}$ s.t.

$$T_A = -S T_A^* S^{-1} = -S T_A^T S^{-1}.$$

$$S = S^{-1} = \prod_{a \text{ odd}} Y_a \prod_{b \text{ even}} X_b$$

$$T_{a,2n+1} = -S\, T^*_{a,2n+1} S. \quad a = 1..n$$

$\Rightarrow$ $\underline{2^n}$ of $SO(2n+1)$ is not complex.

n	S	sym. of S
1	Y_1	AS
2	$Y_1 X_2$	AS
3	$Y_1 X_2 Y_3$	S
4	$Y_1 X_2 Y_3 X_4$	S $\leftarrow$

$\vdots$ repeats mod 4.

n mod 4	$2n+1$ mod 8	G	sym of S	R is
4	1	$SO(8k+1)$	S	real
1	3	$SO(8k+3)$	AS	pseudoreal
2	5	$SO(8k+5)$	AS	pseudo. real
3	7	$SO(8k+7)$	S	real

for $SO(2n)$ $\quad 2^n = 2^{n-1}_+ \oplus 2^{n-1}_-$

eigenspace of γ_F $\quad \gamma_F^2 = 1$

<u>claim</u>: if $[S, \gamma_F] = 0$ $\quad S\, 2^{n-1}_\pm S^{-1} = 2^{n-1}_\pm$ even n

if $\{S, \gamma_F\} = 0$ $\quad S\, 2^{n-1}_\pm S^{-1} = 2^{n-1}_\mp$ odd n

γ_F acts as $\Gamma = \prod_a Z_a$.

$$S \Gamma S^{-1} = (Y_1 X_2 Y_3 X_4 \cdots)(Z_1 Z_2 Z_3 Z_4 \cdots)(Y_1 X_2 Y_3 X_4 \cdots)$$

$Y_1 Z_1 Y_1 = -Y_1^2 Z_1$
$= -Z_1$

$X_2 Z_2 X_2 = -X_2^2 Z_2$
$= -Z_2$

$$= (-1)^n \Gamma.$$

G	n mod 4	2n mod 8	2n+1 mod 8	S	R
so(8k+1)	4		1	sym	real
so(8k+2)	1	2		take R to $\bar{R}$	complex
so(8k+3)	1		3	is AS	pseudoreal
so(8k+4)	2	4		is AS	pseudoreal
so(8k+5)	2		5	is AS	pseudoreal
so(8k+6)	3	6		take R to $\bar{R}$	complex
so(8k+7)	3		7	is Sym	Real
so(8k)	4	8		is Sym	two Real

repeats mod 8.
("Bott periodicity".)

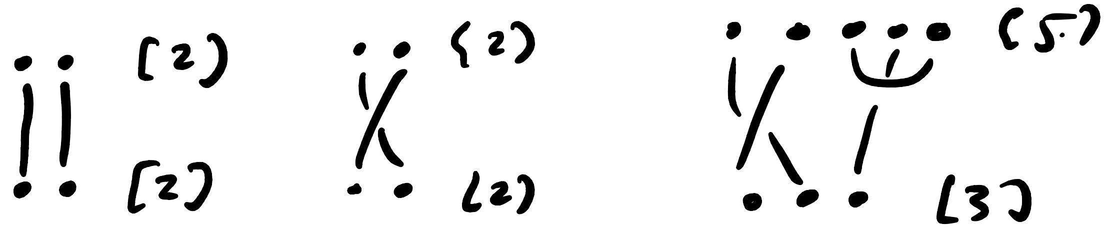
$$S D(g) S^{-1} = D^*(g) \quad \forall g.$$

4.1 Tensor methods by diagrams

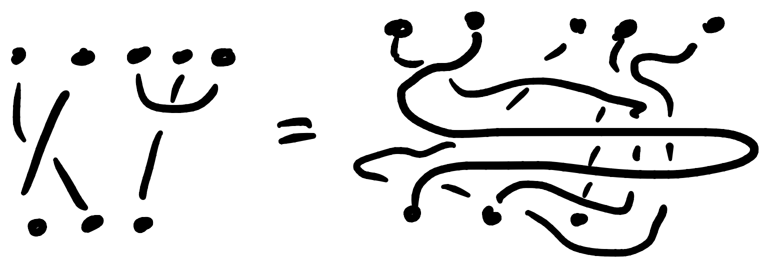
objects: $[0] = \emptyset$ $\quad [1] = \bullet$ $\quad [2] = \bullet\bullet$ $\quad [3] = \bullet\bullet\bullet$ ...

like irreps

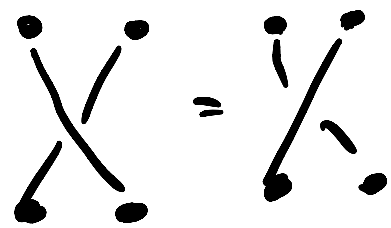
morphisms:



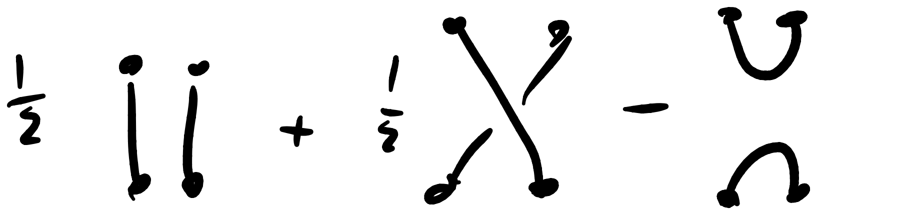
like invariant tensors.



[HERE:

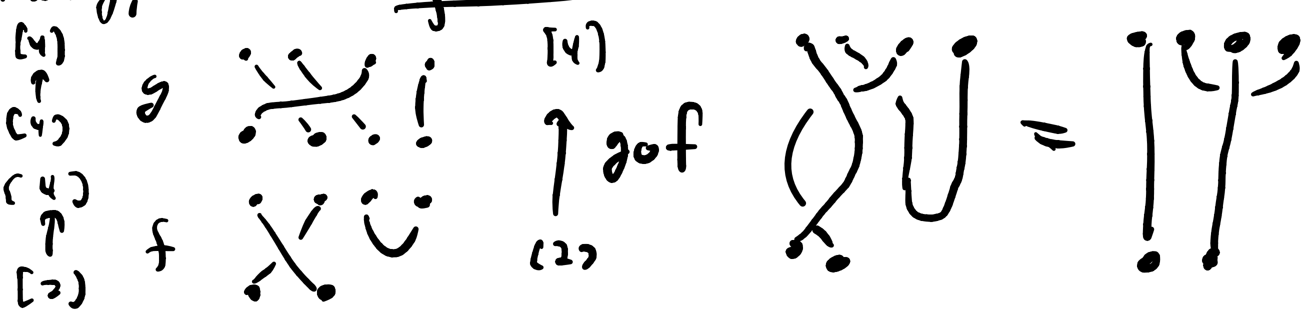


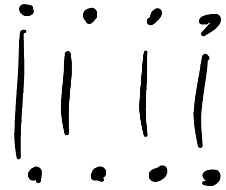
morphisms $[i] \to [j]$ from a vector space over $\mathbb{C}$

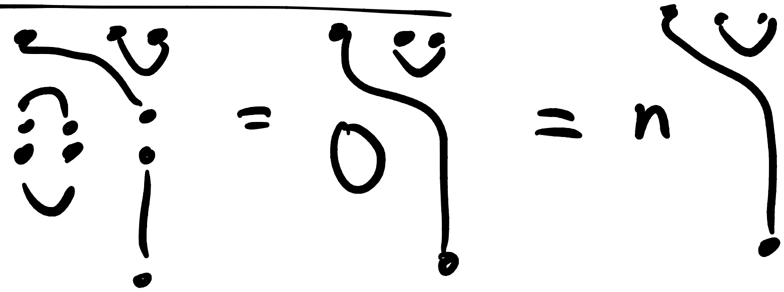


["Brauer algebra"]

actually, it's an algebra:

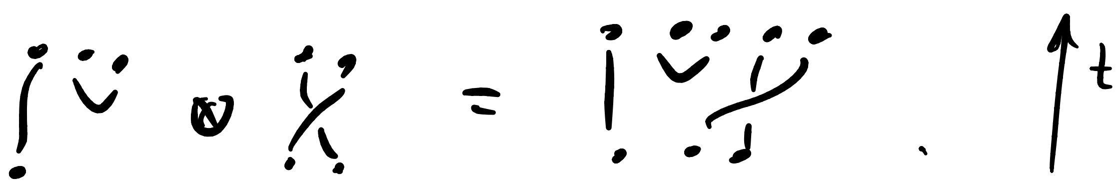


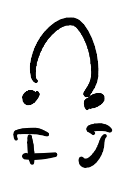
id : [n] → [n] 

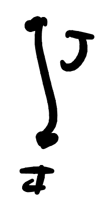
one more rule:  $n \in \mathbb{C}$.

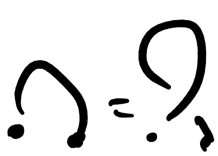
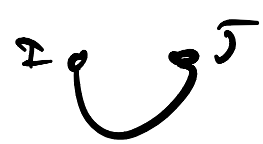
*

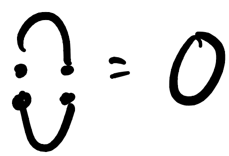
tensor products: $[k] \otimes [l] = [k+l]$



O(n) : preserves $\delta_{IJ} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ 

$\delta_I^J : \mathbb{R}^n \to \mathbb{R}^n$ 

$\delta_{IJ} = \delta_{JI} =$  $\delta^{IJ} : \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}^n$ 

$\delta_{IJ}\delta^{IJ} = \delta_{II} = n =$  *

what's an irrep? what's its dim?

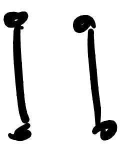
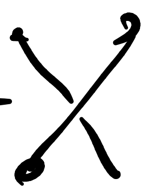
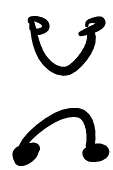
Any unitary rep R (like $R = n^{\otimes k}$)

$$R = \bigoplus_{\text{irreps } a} (\underbrace{a \oplus a \oplus \cdots \oplus a}_{i_a \text{ times}}) = \bigoplus_{\text{irreps } a} \underbrace{V_a}_{(\dim V_a = i_a.)} \otimes a$$

$$\iff \mathbb{1}_R = \sum_{a, i_a} P_{a,i_a} \qquad P_{a,i_a} P_{b,i_b} = \delta_{ab}\delta_{i_a i_b} P_{a,i_a}$$

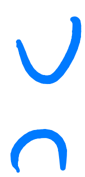
orthogonal projectors.

eg: morphisms $[2) \to [2)$.

$T_1 =$  $T_2 =$  $T_3 =$ 

$$P_1 = T_3/n$$

$$P_1 = c\,T_3 \qquad P_2 = \tfrac{1}{2}(T_1 + T_2) - T_3/n, \quad P_3 = \tfrac{1}{2}(T_1 - T_2)$$

$P_1^2 = c^2$  $= c^2$  $n = c^2 n \cdot T_3 \overset{!}{=} c\,T_3$

$$P_i P_j = \delta_{ij} P_j \quad ij = 1..3. \Rightarrow id_{2\to 2} = P_1 + P_2 + P_3.$$

$$\underline{n}^{\otimes 2} = 1 \oplus \mathrm{Sym}^2_{\text{traceless}}\,\underline{n} \oplus \Lambda^2 \underline{n} = R_1 \oplus R_2 \oplus R_3.$$

(eg $n=3$ $\qquad 1 \oplus 5 \oplus 3$)

<u>dims?</u> $\qquad \dim R_a = \operatorname{tr} P_a$

$f : [k) \to [k) \qquad \operatorname{tr} f =$ 

← k lines

← k lines

$$\dim R_1 = \operatorname{tr} P_1 = \frac{1}{n}\left(\begin{smallmatrix}\smile\\ \frown\end{smallmatrix}\right) = \frac{1}{n}\,\bigcirc = 1.$$

$$\dim R_2 = \operatorname{tr} P_2 = \frac{1}{2}\Big(\big(\,|\,|\,\big) + \big(\times\big)\Big) - \left(\begin{smallmatrix}\smile\\ \frown\end{smallmatrix}\right)/n$$

$$= \frac{1}{2}(n^2+n) - 1$$

$$\dim R_3 = \operatorname{tr} P_3 = \frac{1}{2}\Big(\big(\,|\,|\,\big) - \big(\times\big)\Big)$$

$$= \frac{1}{2}\,n(n-1).$$

$\exists !$ decomposition $\forall k \in \mathbb{Z}_{\geq 0}$, $\underline{\forall n \in \mathbb{C}}$.

$$\mathrm{id}_{[k) \to [k)} = \sum_a P_a \qquad\qquad \dim R_a = \operatorname{tr} P_a.$$

$\exists$ values of n s.t. $\dim R_a = 0$.

eg: $\operatorname{tr} P_3 = \frac{n(n-1)}{2} = 0$ if $n = 0, 1$

$\operatorname{tr} P_2 = \frac{(n+2)(n-1)}{2} = 0$ if $n = 1$ or $n = -2$.

(claim: this only happens for $n \in \mathbb{Z}$.)

Motivation for $n \in \mathbb{R} \setminus \mathbb{Z}$: $S_x^i S_y^j \delta_{ij}$ is $O(n)$ inv't.

O(n) model

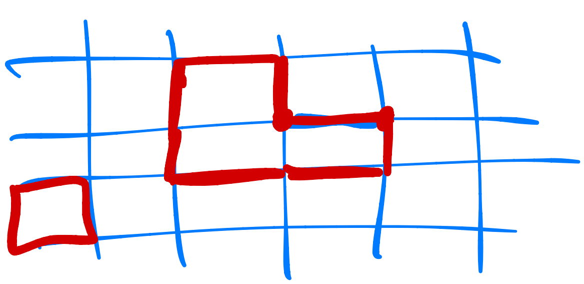
$$Z_{O(n)} = \prod_x \int ds_x^i \, e^{-\beta \sum_{\langle x,y\rangle} \vec{S}_x \cdot \vec{S}_y} \qquad \left(\sum_i S_x^i S_x^i = 1.\right)$$

$$\simeq \prod_x \int ds_x^i \prod_{\langle xy\rangle} \left(1 + \sum_i K S_x^i S_y^i\right)$$

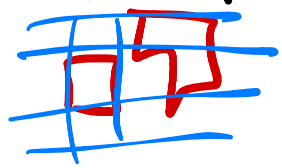
universal bits.

Integration table: $\int ds_x \, 1 = 1 \qquad \int ds_x \, S_x^i = 0$

$$\int ds_x (S_x^i)^2 = \frac{1}{6\pi}.$$



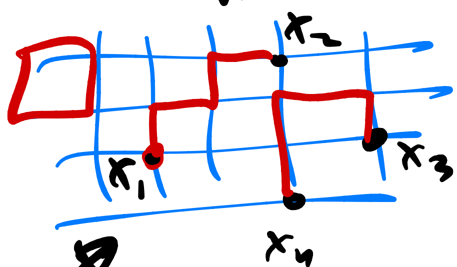
$$Z_{O(n)} = \sum_{\substack{\text{collections of} \\ \text{closed loops}}} \tilde{K}^{\text{length of loops}} \, n^{\#\text{ of components}}$$



n need not be $\in \mathbb{Z}_{\geq 0}$.

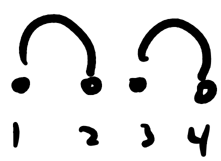
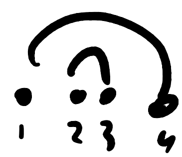
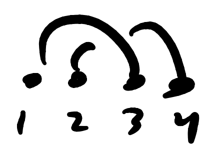
$$\langle S_{x_1} S_{x_2} S_{x_3} S_{x_4} \rangle = \sum_{\substack{\text{collection of} \\ \text{loops}}} n^{\#\text{ of components}} \, \tilde{K}^{\text{length}}$$

$O(n)$



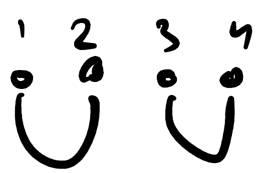
$n \to 0$ forbids loops!

$$Z_{O(n \to 0)} = Z_{\substack{\text{self-avoiding} \\ \text{walks}}}$$

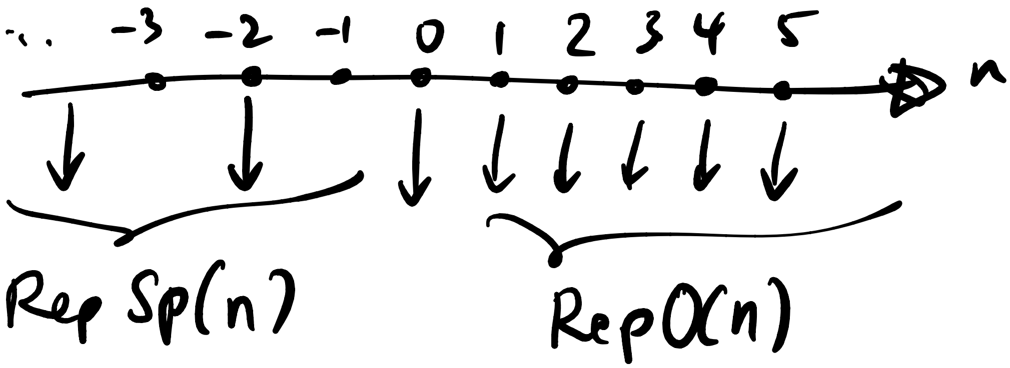
$c_1(x_1 \ldots x_4)$  $+ \; c_2(x_1 \ldots x_4)$  $+ \; c_3(x_1, x_4)$ 

$\in$ (morphisms: $[4] \to [0]$.)

to get a #: $\circ$ mor $[0] \to [4]$.

$\langle S_{x_1} \cdots S_{x_4} \rangle \circ$  $= n^2 c_1 + n c_2 + n c_3$.

(empty diagram)



$n \longleftrightarrow -n$

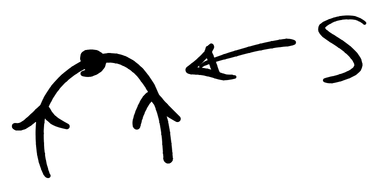
$\begin{smallmatrix}\square\\\square\end{smallmatrix} \longleftrightarrow \square\square$

$\dfrac{n(n-1)}{2} \longleftrightarrow \dfrac{n(n+1)}{2}$

Sym $\longleftrightarrow$ Λ

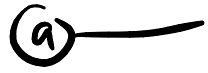
$Sp(n)$ preserves ω $\longleftrightarrow$ $O(n)$ preserves δ.

<u>Generalization</u> : $\big|$ $\leftrightarrow$ Rep R of G

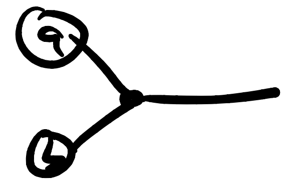
if R is real : $\exists$ a invariant tensor 

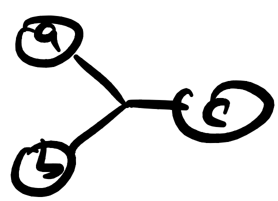
$\Rightarrow$ no arrow.

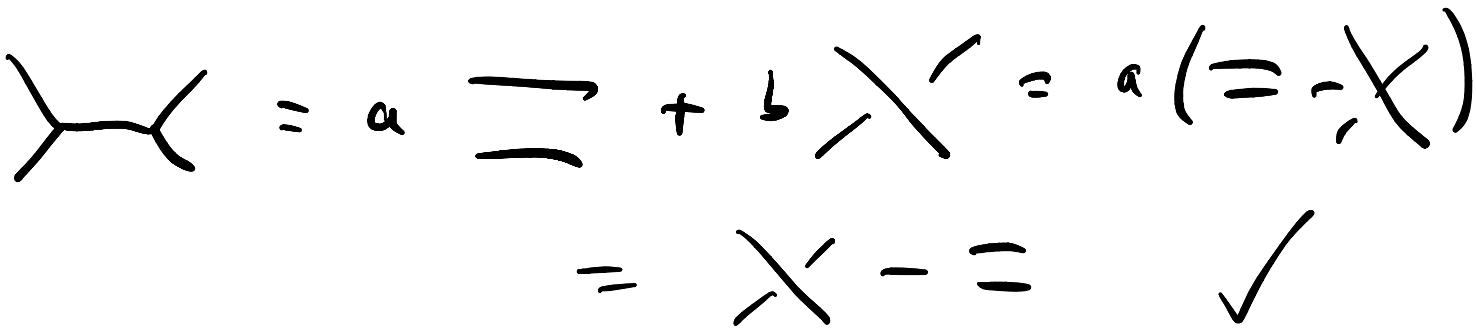
ϵ. O(n) w/ $n = 3$.

$(\vec{a})_i = a_i$ 

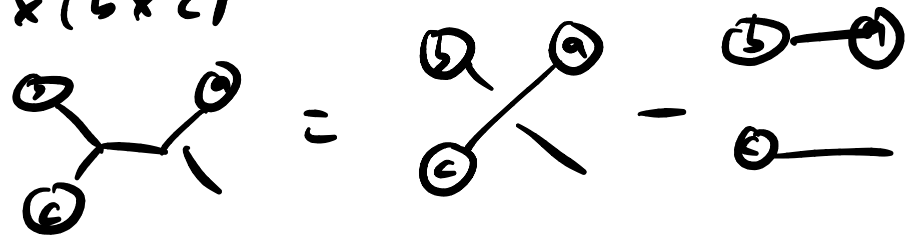
$\vec{a}\cdot\vec{b} = \sum_i a_i b_i$ 

$(\vec{a}\times\vec{b})_i = \epsilon_{ijk}\, a_j b_k =$ 

$(a\times b)\cdot c =$  is cyclically sym.

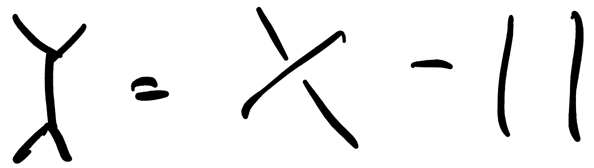


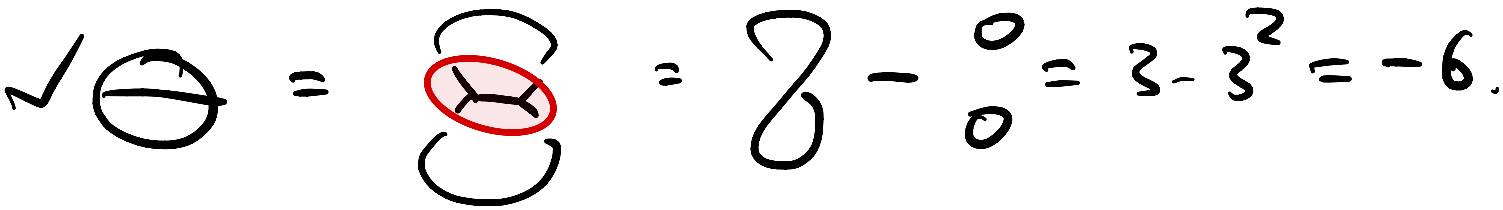
$a\times(b\times c)$

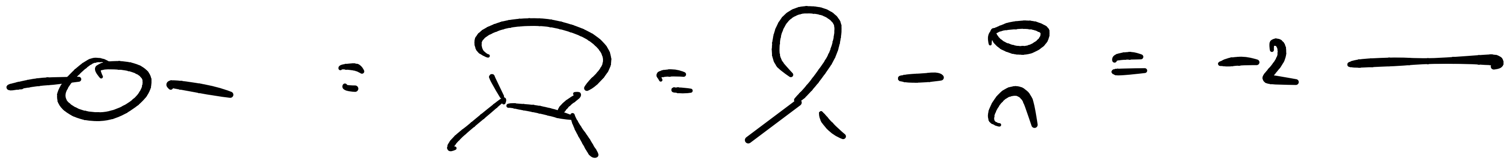


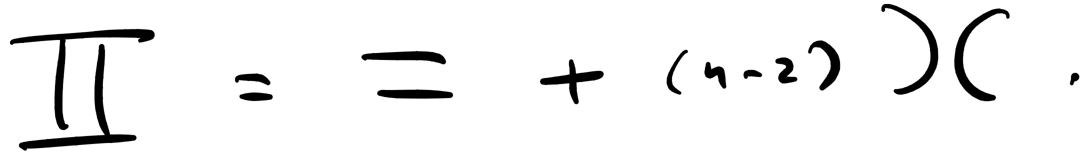
$= b\,(a\cdot c) - c\,(\cancel{d}a\cdot b)$

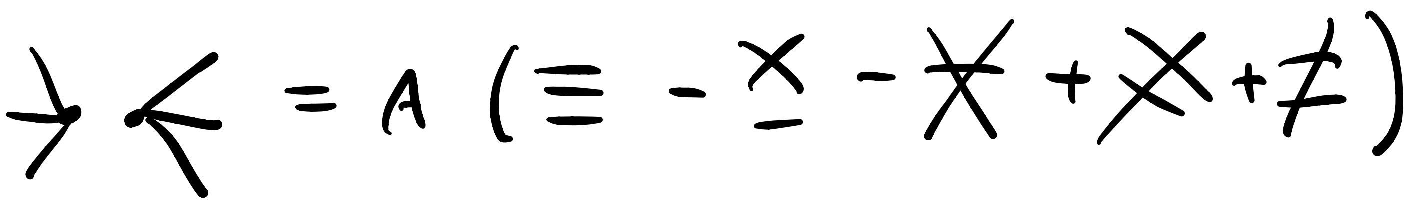
$\epsilon_{ijm}\,\epsilon_{klm} = \delta_{ik}\,\delta_{jl} - \delta_{il}\,\delta_{jk}$

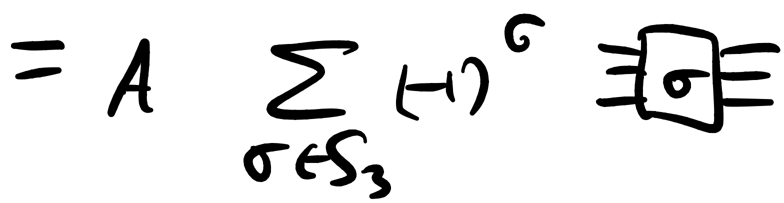


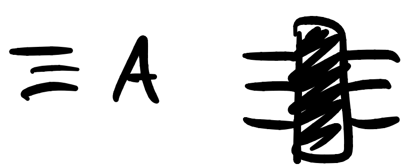




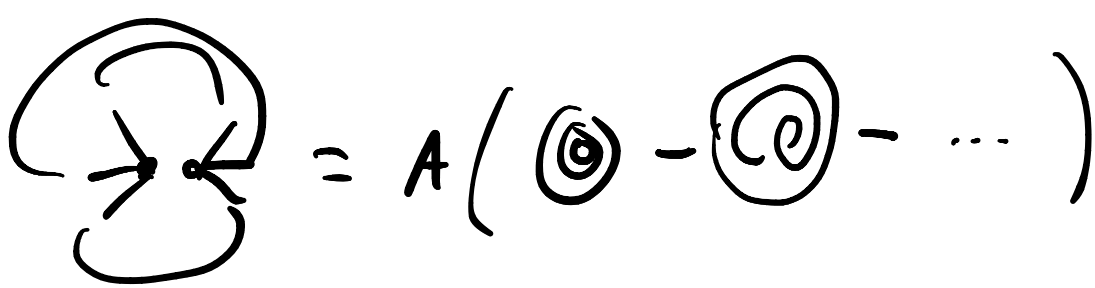




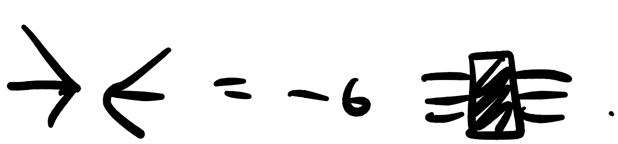




tr(RHS):



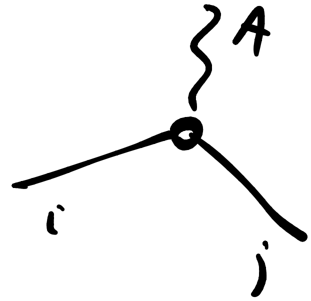
$= -6 \qquad = A(n^3 - 3n^2 + 2n) = A \cdot 6$



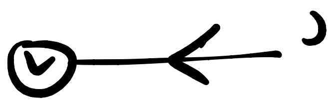
$(J^i)_{jk} = -i\,\epsilon_{ijk}$. generators of $SO(3)$

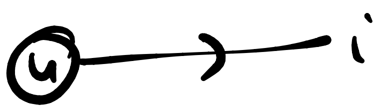
which generator.

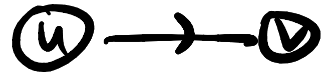
which index in 3

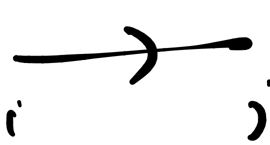
$(T^A)_{ij} =$  $]$

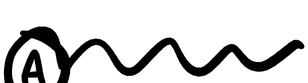
SU(n):

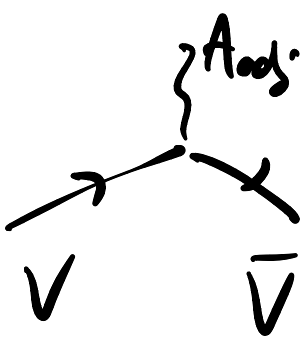
 $v^j \in \underline{n}$

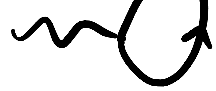
 $u_i \in \bar{n}$

$u \cdot v = u_k\, \delta^k_j\, v^j =$  .

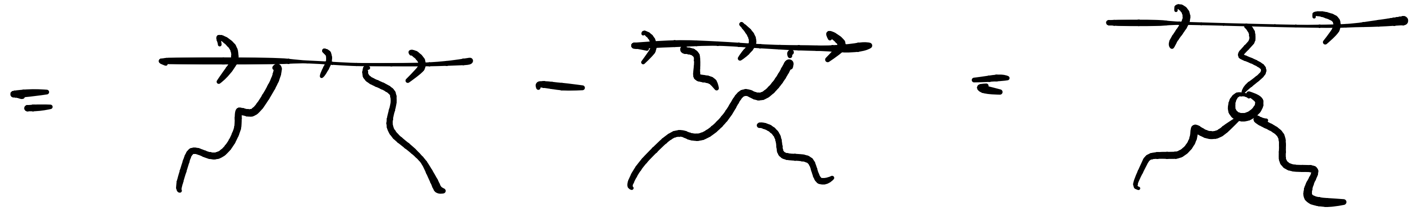
 $= \delta^i_j$ is an inv't tensor.

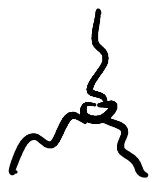
 $\in$ adjoint $=$ $\overline{\text{adjoint}}$.

 (T^A) in the V rep.

$\mathrm{tr}\, T^A = (T^A)^i_i =$  $= 0$.

$$[T^A, T^B]^j{}_k = i f^{ABC} (T^C)^j{}_k$$



 $\in \mathrm{adj}^3$

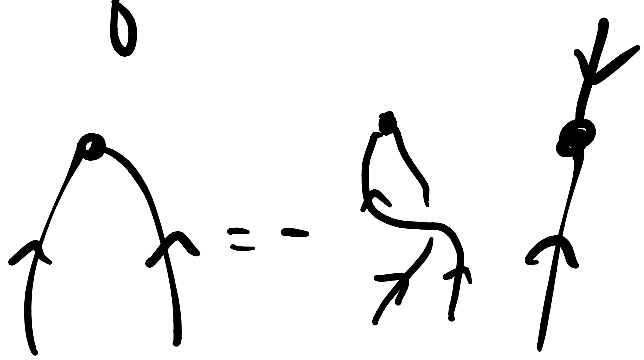
"birdtracks".

$\underline{3}$ of $SO(3) = SU(2)$ is real $3 = \bar{3}$.

fund of $SO(3)$ = adj of $SU(2)$

= adj of $SO(3)$

$\underline{2}$ of $SU(2)$ is pseudoreal.



$2 \rightarrow \bar{2}$.

$2 \otimes 2 \rightarrow$.

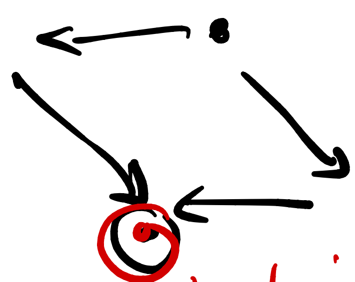
$$\epsilon_{ijk} R^i{}_l R^j{}_m R^k{}_n = \det R \, \epsilon_{lmn}.$$

$$R \in O(3) \setminus SU(3)$$

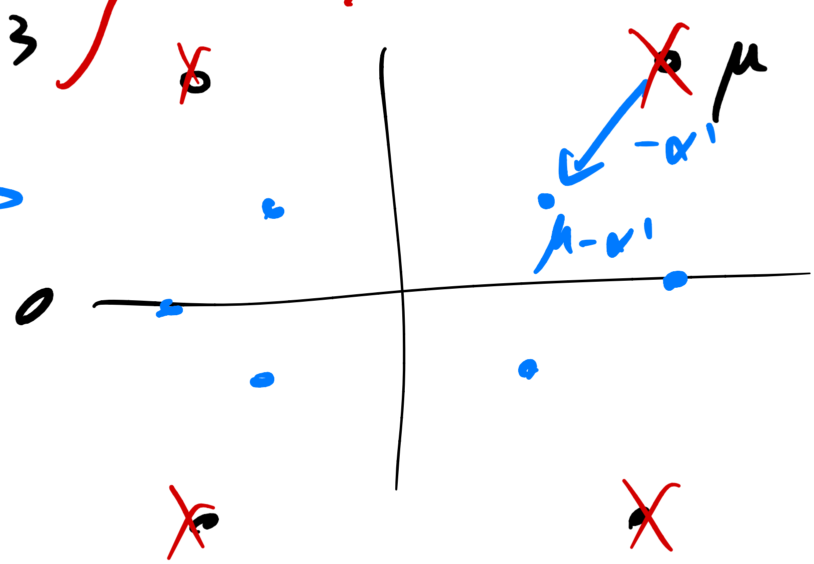
Cvitanovic.

Peeling: In an irrep multiplicity of HW = 1.

If μ is the HW of an irrep we peel it off.



eg: HW of $V^{\otimes 3}$ $= R_\mu \oplus$ (whatever's left)



orbit of μ under Weyl

$$\chi_R(g) = \mathrm{tr}_R\, U(g) = \mathrm{tr}_R\, e^{i\theta_a H_a} = \sum_\mu n_\mu e^{i\theta_a \mu_a}$$