3.6 Classification of Simple Lie algebras

The simple roots of a <u>simple</u> Lie alg $\mathfrak{g}$ satisfy:

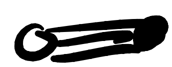
A) there are $\mathrm{rank}(\mathfrak{g}) = r$ of them, lin. indep.

B) $A_{\alpha\beta} = \dfrac{2\alpha\cdot\beta}{\alpha^2} \in \{0, -1, -2, -3\}$

C) Dynkin diagram is Connected.

<u>Lemma 1</u>: A 3-node connected subdiagram must be:

$\Rightarrow$ the only triple line occurs in G_2



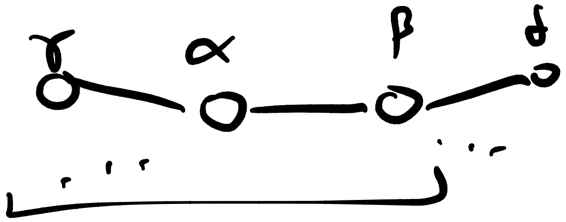
<u>Lemma 2</u>: If $\cdots \overset{\alpha}{\circ}\!-\!\!\!-\overset{\beta}{\circ}\cdots$ is ok

then so is $\cdots \underset{\alpha+\beta}{\circ} \cdots$ is too.

<u>Pf</u>: Recall if $\overset{\alpha}{\circ}\!-\!\!\!-\overset{\beta}{\circ}$ ie $\alpha^2 = \beta^2$ and $\dfrac{\alpha\cdot\beta}{\alpha^2} = -\dfrac{1}{2}$

then $\alpha+\beta$ is also a root.

$\Rightarrow$ this is a <u>subalgebra</u>



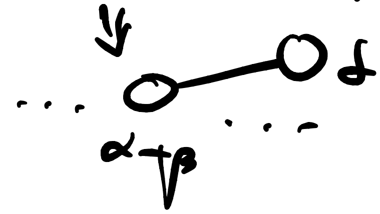
$$\left.\begin{aligned}\gamma\cdot\beta&=0\\ \delta\cdot\alpha&=0\end{aligned}\right\} \longleftarrow \text{lemma 1.}$$

$$\alpha^2=\beta^2=0.$$

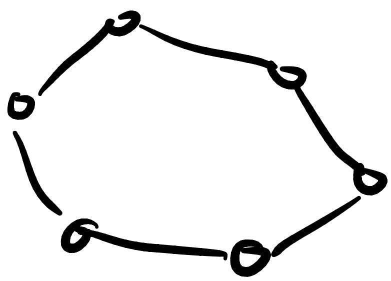
$$\Rightarrow (\alpha+\beta)^2=\alpha^2+\beta^2+2\alpha\cdot\beta$$
$$=\alpha^2.$$

$$\Rightarrow \gamma\cdot(\alpha+\beta)=\gamma\cdot\alpha \qquad \delta\cdot(\gamma+\beta)=\delta\cdot\beta$$

$\Downarrow$ 

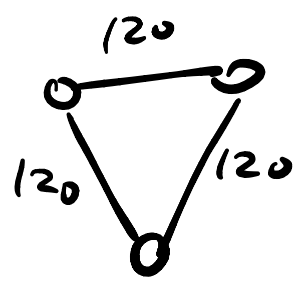
$\Downarrow$ 

lemma 2 $\Rightarrow$



X

smoosh $\longrightarrow$



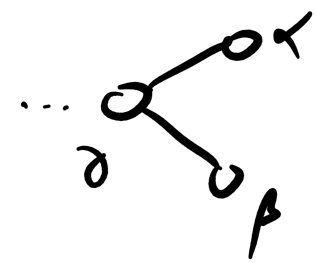
lemma 1 X

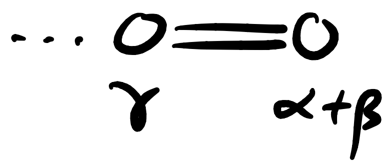
<u>no loops!</u>

X

smoosh $\longrightarrow$

X lemma 1

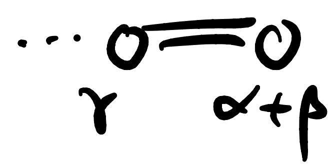
Lemma 3: If  is OK

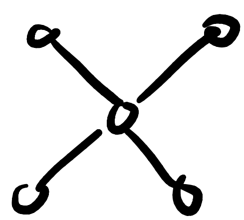
then  is too.

Pf: $\underline{\alpha\cdot\beta=0}$. $(\alpha+\beta)^2=\alpha^2+\beta^2=2\alpha^2$. $\alpha^2=\gamma^2=\beta^2$

$$\frac{2\alpha\cdot\gamma}{\alpha^2}=\frac{2\alpha\cdot\gamma}{\gamma^2}=\frac{2\beta\cdot\gamma}{\beta^2}=\frac{2\beta\cdot\gamma}{\gamma^2}=-1.$$

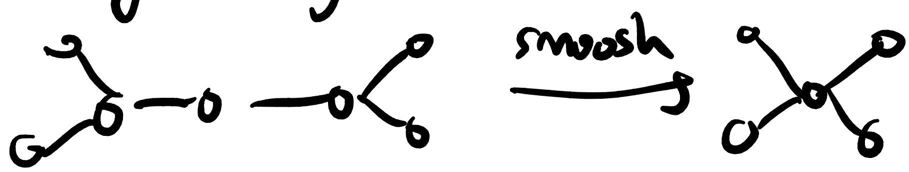
$$\Rightarrow \frac{2(\alpha+\beta)\cdot\gamma}{\gamma^2}=-2 \qquad \frac{2(\alpha+\beta)\cdot\gamma}{(\alpha+\beta)^2}=-1.$$

$\Rightarrow$  $\square$

Lemma 3 $\Rightarrow$ if  were ok, then  would be ✗ by lemma 1.

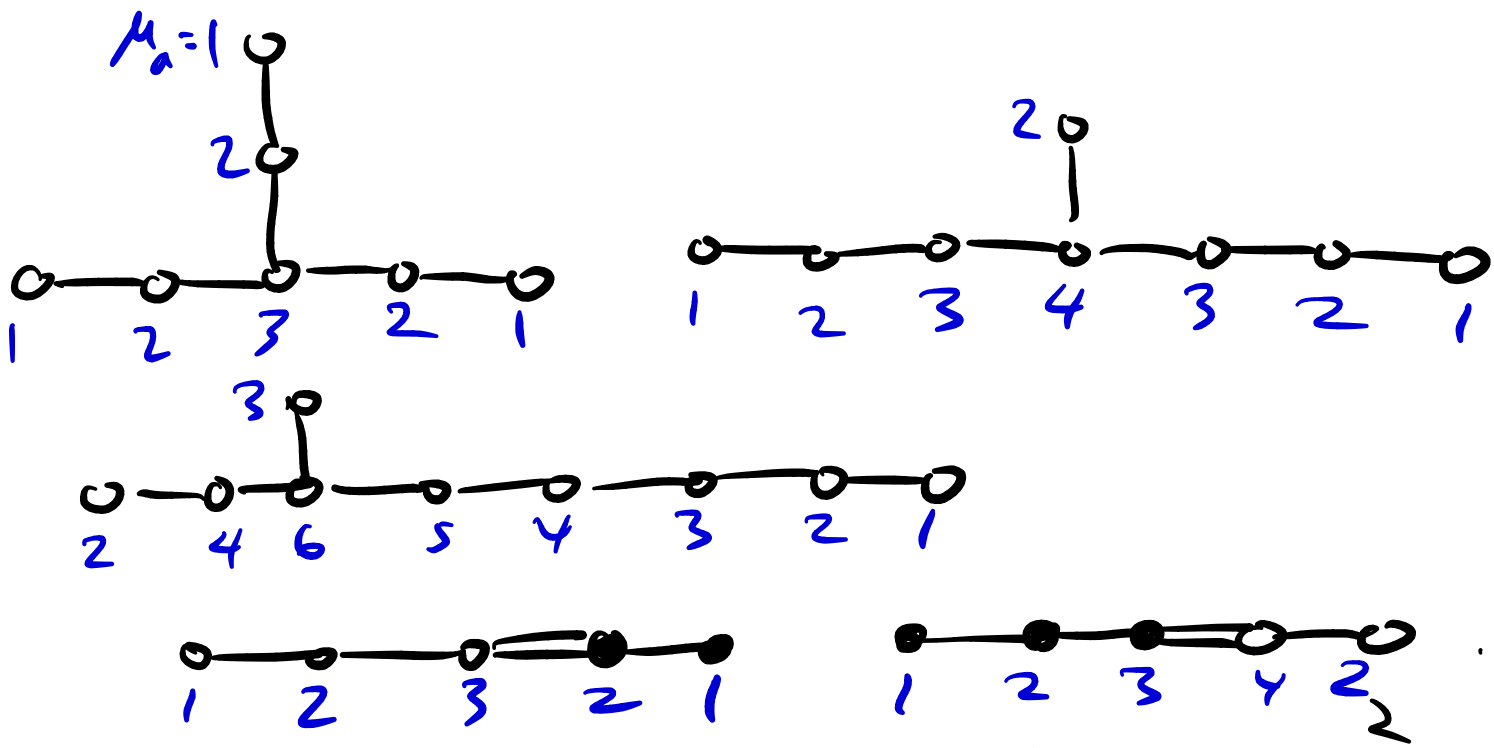
✗

$\Rightarrow$ only one junction is allowed



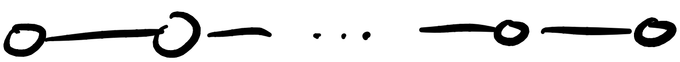
Lemmas 1 & 3: ≤ 3 lines can come out of any node.

Lemma 4: The following can't satisfy (A):

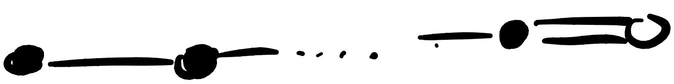


Pf: $\sum_{a=1}^{r} \mu_a \alpha_a = 0$. $\Leftarrow$ $\left(\sum \mu_a \alpha_a\right) = 0$

This leaves:

 $A_n = su(n+1)$

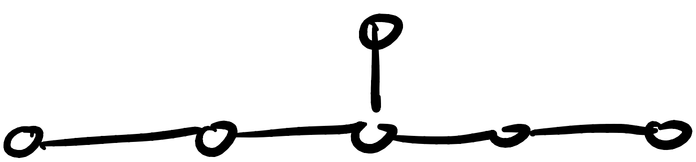
 $B_n = so(2n+1)$

 $C_n = sp(2n)$

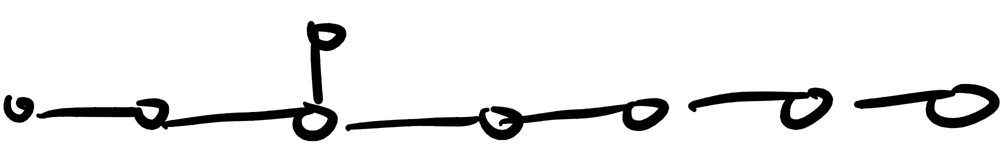
 $D_n = so(2n)$

 G_2

 F_4

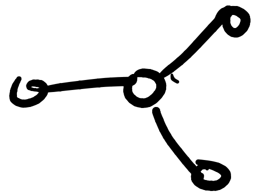
 E_6

 E_7

 E_8

Coincidences at low rank $A_1 = B_1 = C_1 =$ "o"

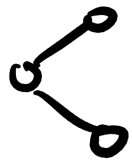
$B_2 = C_2$  $so(5) = sp(4)$.



D_4 has S_3 symmetry.

"triality".

$\downarrow$

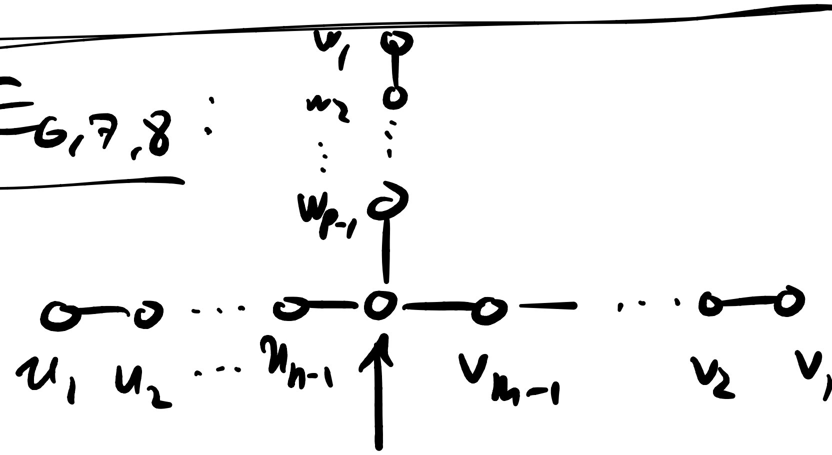
 $= D_3 = A_3 =$ 

$\downarrow$

 $= D_2 = A_1 \times A_1$ $\qquad SO(4) = SU(2) \times SU(2)$.

(D_1 does not exist)

How to discover $E_{6,7,8}$:



$x = w_p = v_m = u_n$

$\alpha^2 = 1$.

Let:

$$\begin{cases} u \equiv \sum_{k=1}^{n-1} k u_k \\ v = \sum_{k=1}^{m-1} k v_k \\ w = \sum_{k=1}^{p-1} k w_p. \end{cases} \qquad \begin{cases} u \cdot v = 0, \; v \cdot w = 0, \; u \cdot w = 0. \\ u^2 = \frac{1}{2} n(n-1), \; v^2 = \frac{1}{2} m(m-1), \; w^2 = \frac{1}{2} p(p-1). \\ x \cdot u = -\frac{1}{2}(n-1), \; x \cdot v = -\frac{1}{2}(m-1) \; x \cdot w = -\frac{1}{2}(p-1). \end{cases}$$

let $s \equiv x - (x\cdot\hat{u})\hat{u} - (x\cdot\hat{v})\hat{v} - (x\cdot\hat{w})\hat{w}$.

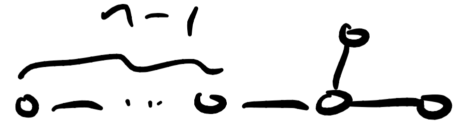
$\left(\hat{u} = \frac{u}{\sqrt{u^2}} \ldots\right)$

$$0 < s^2 = \frac{3}{2} - \frac{1}{2}\left((x\cdot\hat{u})^2 + (x\cdot\hat{v})^2 + (x\cdot\hat{w})^2 \right)$$

$$\Longleftrightarrow \quad \boxed{1 < \frac{1}{n} + \frac{1}{m} + \frac{1}{p}}$$

$\Rightarrow$ at least one of (n, m, p) is < 3.

Besides $(n, m, p) = (n, 2, 2)$

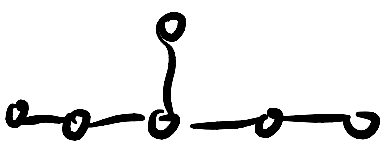


$n-1$

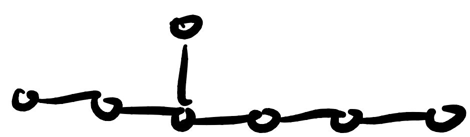
$= SO(2n+4)$ ✓

the only sol'ns are

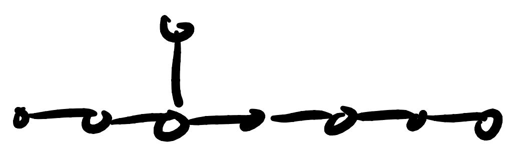
$(3, 3, 2)$ $\quad (3, 4, 2)$ $\quad (3, 5, 2)$



E_6



E_7



E_8.

Dynkin diagram w/ no double or triple lines = simply laced.

= ADE.

eg: Discrete subgroups of $SU(2)$ or $SO(3)$:

$A_n = \mathbb{Z}_n$

$D_n = D_n$

$E = \{T, O, I\}$.

singularities of complex surface.

$X =$ ADE Dynkin diagram

$\Gamma_X \subset SU(2)$.

$M_X = \mathbb{C}^2/\Gamma_X$ is a space w/ holonomy Γ_X.

string theory on $M_X \to X$ gauge theory.

$H_i \rightsquigarrow$ RR vectors — perturbative strings.

$E_\alpha \rightsquigarrow$ D2 branes

3.7 Classical groups.

$H = H^\dagger \Rightarrow$

$U = e^{iH}$ is unitary

$\operatorname{tr} H = 0 \Rightarrow$ is special unitary $\det U = 1$

dim: $A_{n-1} = SU(n)$.

$SU(n) = \{n\times n,$ traceless, hermitian matrices$\}$. $\left(\frac{n^2-1}{= \dim SU(n)}\right)$

rank is $n-1$.

Cartan subalgebra $= \{a_i h_i \mid \sum_i a_i = 0\}$.

$$h_i \equiv \begin{array}{cc} i & \begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & & 1 & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix} \end{array} \quad \text{i.e. } (h_i)_{jk} = \delta_{ij}\delta_{ik}.$$

$$E_{(ij)} \equiv \begin{array}{cc} & \begin{array}{ccc} & & j \end{array} \\ i & \begin{pmatrix} & & \\ & & 1 \\ & & \end{pmatrix} \end{array} \quad \text{i.e. } (E_{(ij)})_{kl} = \delta_{ik}\delta_{jl}.$$

are eigenvectors of ad_h :

$$[a\cdot h,\ E_{(ij)}] = (a_i - a_j)\,E_{(ij)}$$

$$(a\cdot h) = \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_n \end{pmatrix} \qquad \text{coeff. of } a_l :$$

$$[h_l,\ E_{(ij)}] = (\delta_{li} - \delta_{lj})\,E_{(ij)}$$

$$= (e_i - e_j)_l\ E_{(ij)}$$

$$\left(e_i \equiv \begin{array}{cc} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} & i \end{array} \qquad (e_i)_j \equiv \delta_{ij}. \right)$$

$$\Rightarrow \text{roots are } \{ e_i - e_j \quad i \neq j \}$$

$$\dim su(n) = \underbrace{n-1}_{\text{cartan}} + \underbrace{\frac{n(n-1)}{2}\times 2}_{\text{nonzero roots}} = n^2 - 1.$$

Positive roots: $e^i - e^j \quad i<j \qquad (e^1 > e^2)$

α_{ij}

the roots $e^i - e^j$ are n-component vectors but lie in the plane $\sum_{i,j} \alpha_{ij} = 0$

$\perp (e_1 + e_2 + \dots + e_n)$

Simple roots: $e^1 - e^2, \; e^2 - e^3 \dots \; e^{n-1} - e^n$

$$e^i - e^{i+1}, \; i = 1..n-1.$$

$$A_{k,k+1} = \frac{2\alpha_k \cdot \alpha_{k+1}}{\alpha_k^2} = \frac{2}{2}(e_k - e_{k+1})(e_{k+1} - e_{k+2})$$

$$(A_{k,j\neq k\pm 1} = 0.) \qquad = -1$$

$\Rightarrow \quad \circ - \circ - \circ \; \cdots \; - \circ - \circ \; . \quad \checkmark$

Fund. wts: $\mu^b \qquad \dfrac{2\alpha^a \cdot \mu^b}{(\alpha^a)^2} = \delta^{ab} \qquad$ (α^a simple roots)

claim: $\mu^b = \sum_{a=1}^{b} e^a$ $\qquad$ **why?**

consider $\mu^1 = e^1$. In the $\underline{n}$, fundamental rep,

the evecs of h_i are $|j\rangle$ $j = 1 \ldots n$

w/ evals $\delta_{ij} = (e_i)_j$

$|1\rangle$ is the H.W. state.

$\Rightarrow$ H.W is e_1.

$$\Lambda^m \underline{n} \subset_{AS} \underbrace{\underline{n} \otimes \underline{n} \ldots \otimes n}_{m \text{ times}}$$

H.W. state in $\Lambda^m \underline{n}$ is $|1\rangle \otimes |2\rangle \otimes |3\rangle \ldots \otimes |m\rangle$

$- |2\rangle \otimes |1\rangle \otimes |3\rangle \cdots$

$+ |2\rangle \otimes |3\rangle \otimes |1\rangle$

$+ \ldots$

has weight

$$\sum_{a=1}^{m} e^a = \mu^m \checkmark$$

A basis for $\Lambda^m \underline{n}$ is $|i_1 \ldots i_m\rangle = |i_1\rangle \otimes |i_2\rangle \otimes \cdots$

$- |i_2\rangle \otimes |i_1\rangle \cdots$

$$= \sum_{\pi \in S_m} (-1)^\pi |i_{\pi_1}\rangle \otimes |i_{\pi_2}\rangle \otimes \ldots$$

a general state

$$|A\rangle = \sum_{i_1 \ldots i_m = 1}^{n} A^{i_1 \ldots i_m} |i_1 \ldots i_m\rangle$$

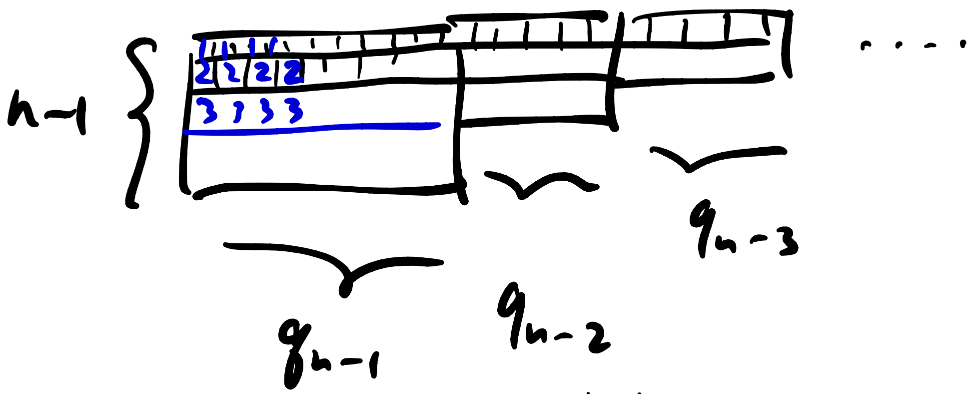
has dim $\binom{n}{m}$

general rep has H.W. $\sum_{m=1}^{r} \underset{\uparrow \, \in \mathbb{Z}_{\geq 0}}{q_m} \underline{\mu}^m$

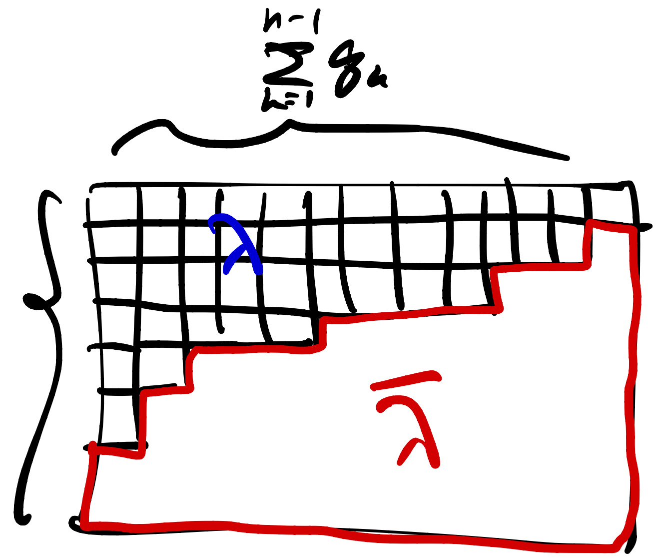
wavef'n has

q_m sets of $\underbrace{m \text{ indices}}_{\text{antisymmetric.}}$

$$
\begin{aligned}
H.W. &= e_1 \left(\sum_{k=1} q_k\right) \\
&+ e_2 \left(\sum_{k=2} q_k\right) \\
&+ e_3 \left(\sum_{k=3} q_k\right) \\
&= \sum_m q_m \mu^m .
\end{aligned}
$$



Rep conjugate to R_λ is $\bar{R}_\lambda = R_{\bar{\lambda}}$



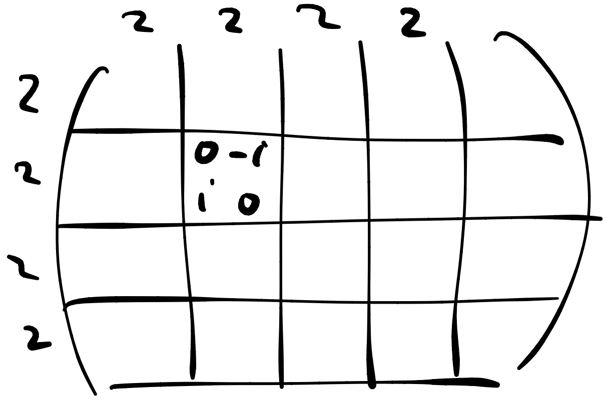
e.g. $\underline{SU(3)}$

$$
\bar{R}_{\square} = R_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}
$$

$$
\bar{R}_{\begin{smallmatrix}\square\square\\\square\square\\\square\end{smallmatrix}} = R_{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}} \quad \}n=3
$$

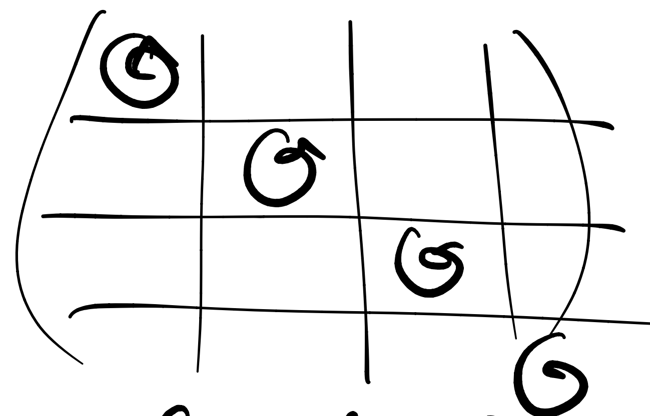
so(N) B_n, D_n.   generators are A.S. (imaginary) $N \times N$ matrices.

$N = 2n$



n 2d subspaces.

cartan generators generate $so(2) < so(2n)$



$$(H_m)_{jk} = -i\left(\delta_{j,2m-1}\delta_{k,2m} - \delta_{j,2m}\delta_{k,2m-1}\right)$$

$$= \sigma^2 \otimes \left(\text{projection onto } m\text{th 2d subspace}\right)$$

has evals ± 1 or 0.

$\Rightarrow$ wts of $2n$ are $\pm e^i$, $i = 1 \ldots n$.

Roots: $\{\pm e^i \pm e^j,\ i \neq j\}$. $\pm$ uncorrelated.

Important note: $\pm 2e^i$ is not a root.

$$E_{2e^i} \, |-e^i\rangle \propto |+e^i\rangle$$

$i=1$.

$$\begin{pmatrix} 1 \\ -i \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ +i \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

no rotation takes this to this (and preserves all other states)

$$\mathrm{diag}(1, -1, \underline{1\ 1\ 1\ 1})$$

has det $= \underline{-1}$. $\Rightarrow$ not a rotation.

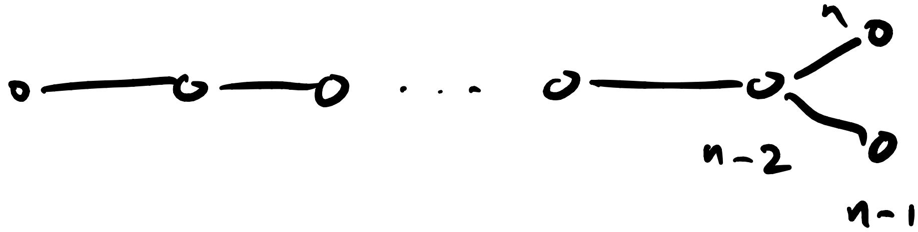
$$n \text{ Cartan generators} + \frac{n(n-1)}{2} \times 4$$

$$= n(2n-1) = \dim SO(2n) \checkmark.$$

positive roots : $\{ e^i \pm e^j \quad i<j \}$

simple roots : $\{ e^1 - e^2, e^2 - e^3 \ldots e^{n-1} - e^n, e^{n-1} + e^n \}$

$= \{ e^i - e^{i+1}, i = 1..n \text{ and } e^{n-1} + e^n \}$.



D_n.

Fund. wts: $\mu^b = \sum_{a=1}^{b} e^a \quad b < n-1$. } HWs of $\Lambda^b \underline{2n}$.

$\mu^{n-1} = \frac{1}{2}(e^1 + \dots e^{n-1} - e^n)$
$\mu^n = \frac{1}{2}(e^1 + \dots e^{n-1} + e^n)$ } HW of spinor reps.

(do not appear in any $\underline{n}^{\otimes k}$.)

$\boxed{N = 2n+1}$ odd.

$$\begin{array}{cccc} 2 & 2 & 2 & 1 \\ \end{array}$$
$$\begin{pmatrix} \circ & & & \\ & \circ & & \\ & & \circ & \\ & & & \end{pmatrix} \ \underline{\text{annihilate}} \ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$
$\to$ wt 0.

Cartan is the same.

new roots which take $\overset{a}{\vee}$ 2d subspace $|e^i\rangle$ to the 1d space of wt 0.

$\pm e^i$.

$\{\pm e^i \pm e^j, \ i \neq j, \ \pm e^i\}$. $\quad \dim SO(2n+1) = n + 4\frac{n(n-1)}{2} + 2n$
$= n(2n+1)$ ✓

<u>positive roots</u>: $e^i \pm e^j \quad i<j, \quad + e^i$.

<u>simple roots</u>: $e^i - e^{i+1} \quad i = 1 .. n-1, \quad e^n$.

short root.

$\Rightarrow$

$$B_n = so(2n+1).$$

fund wts: $\mu^b = \sum_{a=1}^{b} e_a \quad b < n-1 \leftarrow \Lambda^b \, \underline{2n+1}$

and $\mu^n = \frac{1}{2}(e^1 + \dots + e^n)$

$\longleftarrow$ spinor rep.

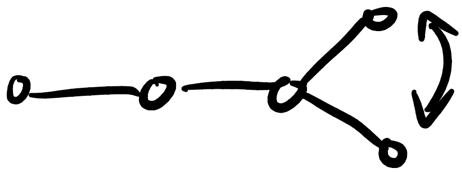
<u>claim</u>: Weyl group = inner automorphisms.

$= N(T)/T$

$T \equiv e^{h} \equiv$ cartan subgroup $= (U(1))^r$.

$$N(T) = \left\{ g h g^{-1} \,\middle|\, \begin{matrix} h \in T \\ g \in G \end{matrix} \right\}$$

symmetries of the Dynkin diagram = outer automorphisms.

eg:  $D_5 = so(10)$

$D_5 / \mathbb{Z}_2 =$  $= B_4 \cdot so(9)$

$D_4 / S_3 =$  G_2