<u>Recap:</u> For each state in any rep of $\mathfrak{g}$

$|\mu\rangle \qquad H_i|\mu\rangle = \mu_i|\mu\rangle$

For each α $\quad (SU(2)_\alpha \subset \mathfrak{g}) \qquad$ $SU(2)_\alpha$ gen. by $\begin{pmatrix} E^\pm, & E^3 \\ \propto E_{\pm\alpha} & \propto \underline{\alpha \cdot H} \end{pmatrix}$

$\exists r \in \mathbb{Z}_{\geq 0} \quad (E^+)^r|\mu\rangle \neq 0$

$$(E^+)^{r+1}|\mu\rangle = 0.$$

$$\Rightarrow \frac{\alpha\cdot\mu}{\alpha^2} + \underline{\underline{r}} = +j.$$

$\exists \ell \in \mathbb{Z}_{\geq 0} \quad (E^-)^\ell|\mu\rangle \neq 0 \quad (E^-)^{\ell+1}|\mu\rangle = 0.$

$$\Rightarrow \frac{\alpha\cdot\mu}{\alpha^2} - \ell = -j$$

$$\Rightarrow \boxed{\frac{2\alpha\cdot\mu}{\alpha^2} = \ell - r} \qquad \frac{\ell - r}{2} = m_z$$

$$\ell + r = 2j$$

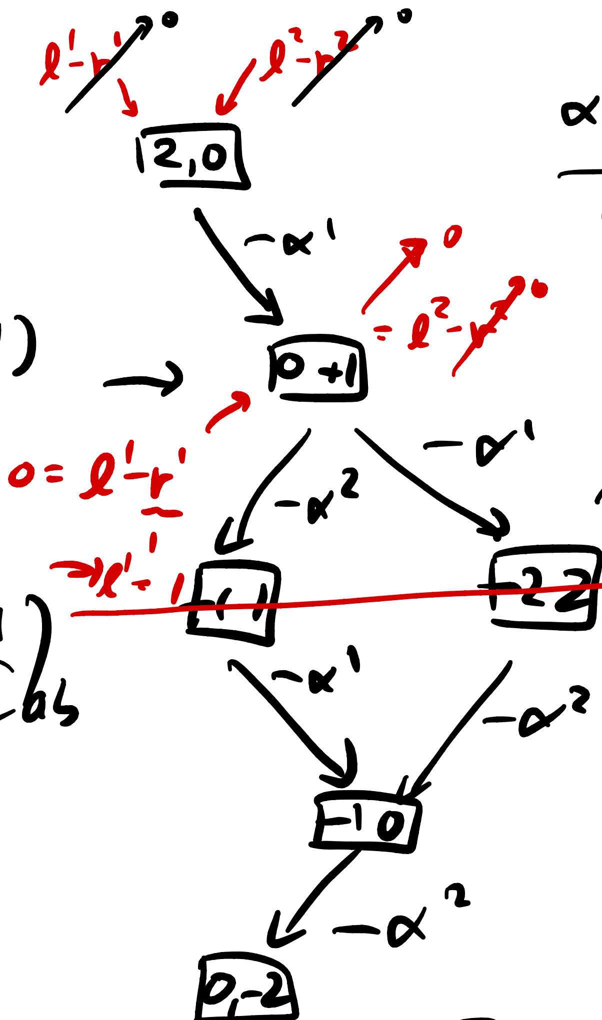
Fundamental weights $\quad \dfrac{2\alpha^a\cdot\mu^b}{(\alpha^a)^2} = \delta^{ab} \quad \begin{pmatrix}\alpha^a \\ \text{simple roots}\end{pmatrix}$

Any highest wt is $\underline{\underline{m_b\mu^b}} \quad m_b \in \mathbb{Z}_{\geq 0}.$

finite dim'l irreps of $\mathfrak{g}$ of rank r $\longleftrightarrow$ $\mathbb{Z}_{\geq 0}^{r}$.

eg: $SU(3)$. $\quad R_{(1,0)} = \underline{3} \quad R_{(0,1)} = \bar{\underline{3}}$.

$\underline{R_{(2,0)} = \,?}$



$$\frac{\alpha^a \cdot (2\mu^1)}{(\alpha^a)^2} = \underline{2\delta^{a,1}}.$$

$$= l^a - r^a.$$

$$l^a - r^a = \frac{\alpha^a \cdot (2\mu^1 - \alpha^1)}{(\alpha^a)^2} \rightarrow$$

$$= \ldots - A_{a1}$$

$$A_{ab} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}_{ab}$$

$$= \frac{\alpha^a \cdot \alpha^b}{(\alpha^a)^2}.$$

$-2 = l^1 - r^1 = l^1 - 2 \Rightarrow l^1 = 0$

$2 = l^2 - \cancel{r^2}$

weyl reflection

$\underline{H^1 \to -H^1}$.

$$|A\rangle = E_{-\alpha_1} \underline{E_{-\alpha_2}} \underline{E_{-\alpha_1} |2\mu^1\rangle} \overset{?}{\propto} \underline{E_{-\alpha_2} E_{-\alpha_1} E_{-\alpha_1} |2\mu^1\rangle} = |B\rangle$$

$$\underbrace{E_{-\alpha_1} E_{-\alpha_2}} = [E_{-\alpha_1}, E_{-\alpha_2}] + E_{-\alpha_2} E_{-\alpha_1}$$

$[E_{-\alpha_1}, E_{-\alpha_2}] \neq 0$, $\propto E_{-\alpha_1 - \alpha_2}$

$$|A\rangle = |B\rangle + [E_{-\alpha_1}, E_{-\alpha_2}] E_{-\alpha_1} |2\mu^1\rangle$$

$[E_\alpha, E_\beta] \propto E_{\alpha+\beta}$

$= 0$ if $\alpha + \beta$ is not a root.

$$= |\beta\rangle + E_{\alpha_1}\underbrace{[E_{-\alpha_1}, E_{-\alpha_2}]}|2\mu'\rangle$$

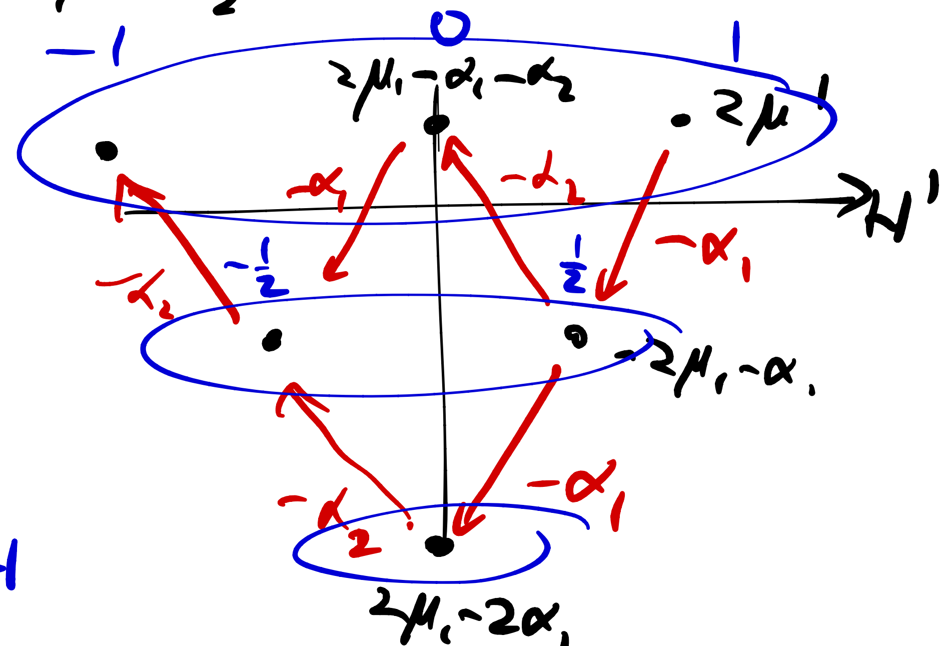
$$E_{-\alpha_2}|2\mu'\rangle = 0$$

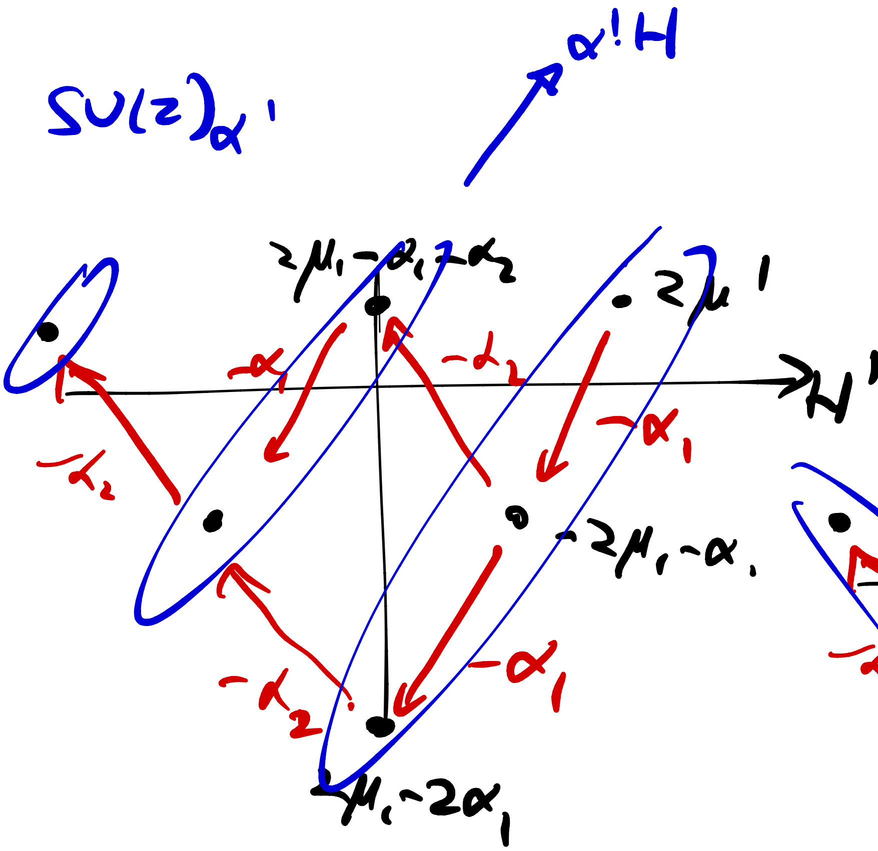
$$= |\beta\rangle + E_{-\alpha_1}E_{-\alpha_1}E_{-\alpha_2}|2\mu'\rangle$$

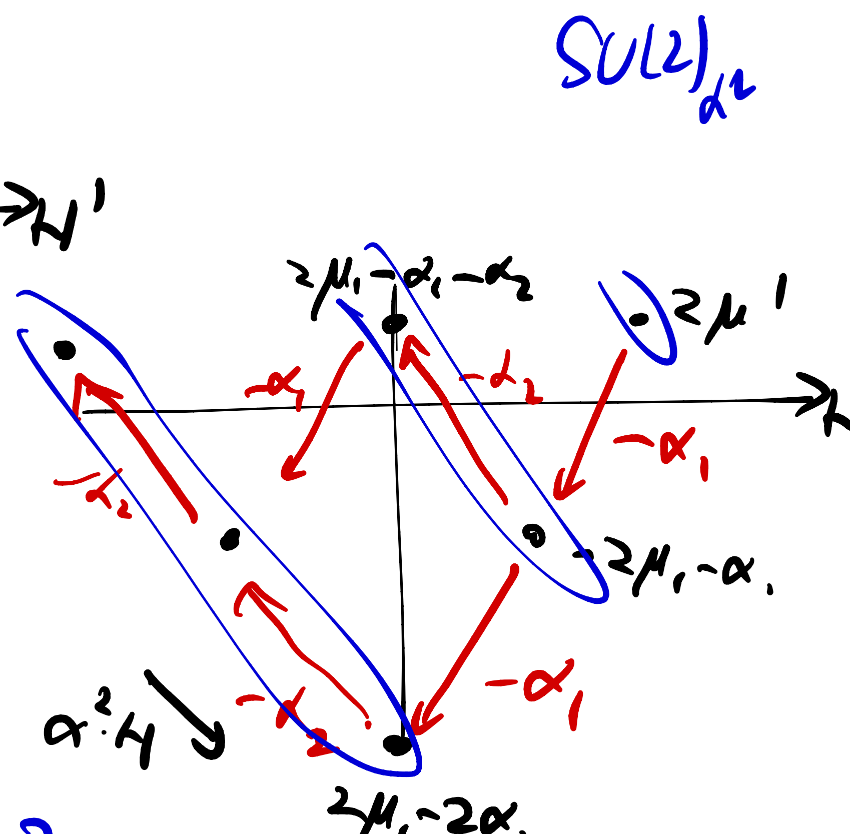
$$= 2|\beta\rangle.$$

$SU(3) \supset SU(2)_{12}$

gen by $T_{1,2,3}$







$3 \oplus 2 \oplus 1 = \underline{6}$

This is the $\underline{6} = \mathrm{Sym}^2\, \underline{3}$

HW of $R_\mu \otimes R_\nu = \underline{\mu + \nu}$. acting on $|\mu\rangle \otimes |\nu\rangle$.

HW.

$R_\mu \otimes R_\nu = \text{span}\{|a\rangle \otimes |b\rangle \}$
$a \in R_\mu,\ b \in R_\nu$

$H_i|\mu\rangle = \mu_i|\mu\rangle \quad \mu_i > a \ \ldots$

acting on $R_\mu \otimes R_\nu$

$\hat{H}_i = H_i \otimes \mathbb{1} + \mathbb{1} \otimes H_i$

HW of $\text{Sym}^2 \underline{3}$?

$\underline{|\mu^1\rangle \otimes |\mu^1\rangle}$ is symmetric

HW of $\underline{\Lambda^2 \underline{3}}$? has wt $\overline{2\mu^1}$.

$|\mu^1\rangle \otimes |\mu^1 - \alpha\rangle - |\mu^1 - \alpha\rangle \otimes |\mu^1\rangle$

2d highest wt.

$\Rightarrow \boxed{HW = 2\mu^1 - \alpha}$.

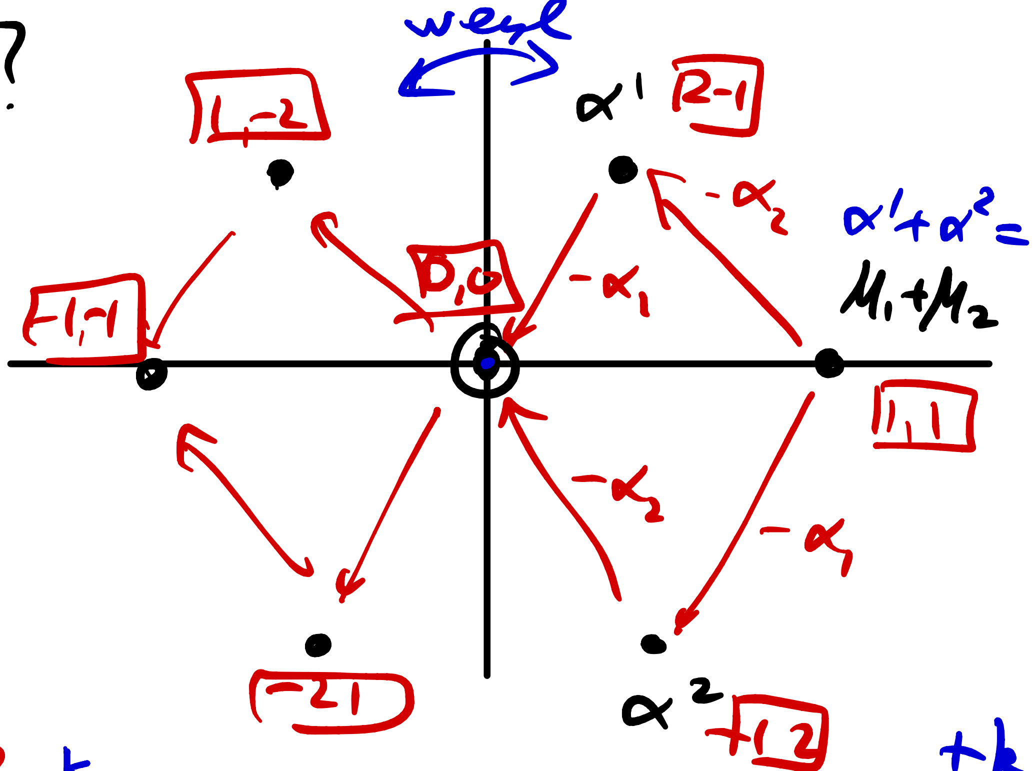
$\boxed{\underline{\text{CLAIM}}: 2\mu^1 - \alpha = \mu^2}$.

$\Rightarrow R_{n\mu^1} = \text{Sym}^n \underline{3}$. $\quad R_{m\mu^2} = \text{Sym}^m \overline{\underline{3}}$.

$R_{\mu^1+\mu^2} = R_{(1,1)}$?

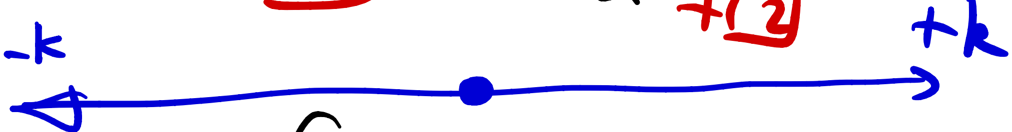
$\mu^1+\mu^2=\alpha^1+\alpha^2$
is a root!

$\Rightarrow$ This is the adjoint!



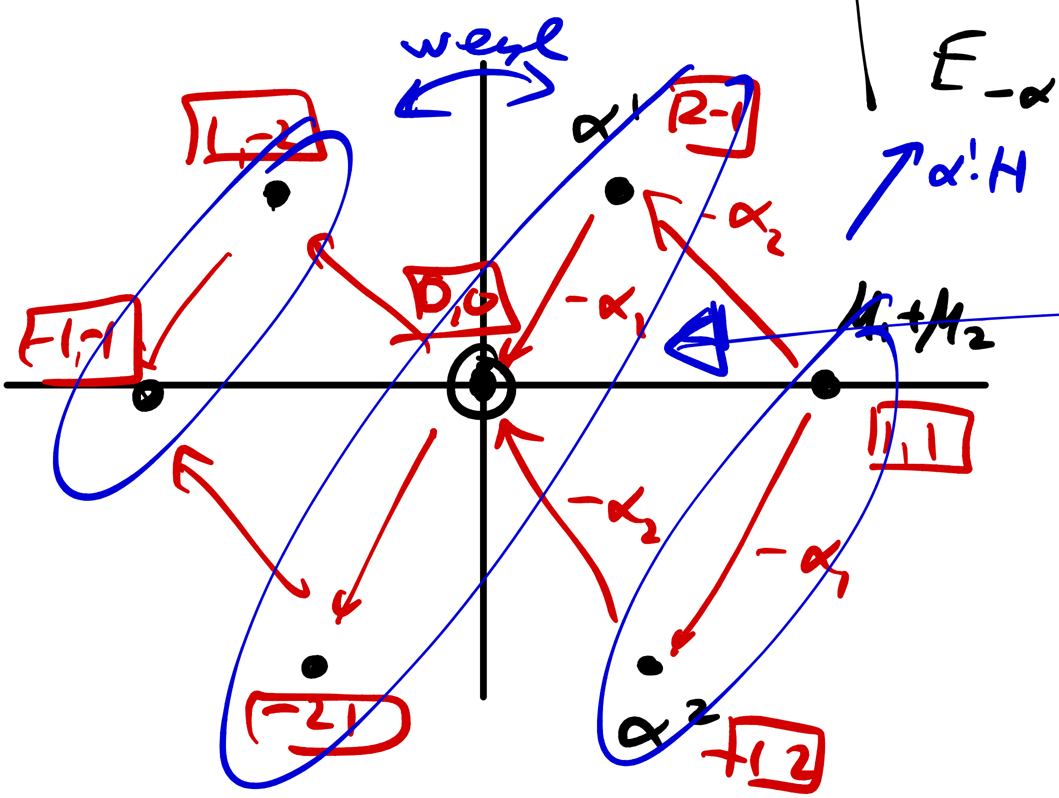
$[H_i, H_j] = 0$

$\Rightarrow H_i|H_j\rangle = 0.$



rank 2 $\Rightarrow$ 2 vectors of wt 0.

$E_{-\alpha_1}E_{-\alpha_2}|\mu_1+\mu_2\rangle$
is lin. indep of
$E_{-\alpha_2}E_{-\alpha_1}|\mu_1+\mu_2\rangle$.



$2\otimes 2$
$=3\oplus 1$

$\underline{8} = \underline{2}\oplus\underline{2}\oplus\underline{3}\oplus\underline{1}$
under $SU(2)_{\alpha_1}$.

Reps of $SU(3)$ $\longleftrightarrow$ {pairs of non-negative integers}
$m\mu^1 + n\mu^2$

$\longleftrightarrow$ m cols of 1 box
n cols of 2 boxes

$\underline{3} = (1,0) = \square$

$\underline{6} = (2,0) = \square\square = \mathrm{Sym}^2 \underline{3}$

$8 = (1,1)$

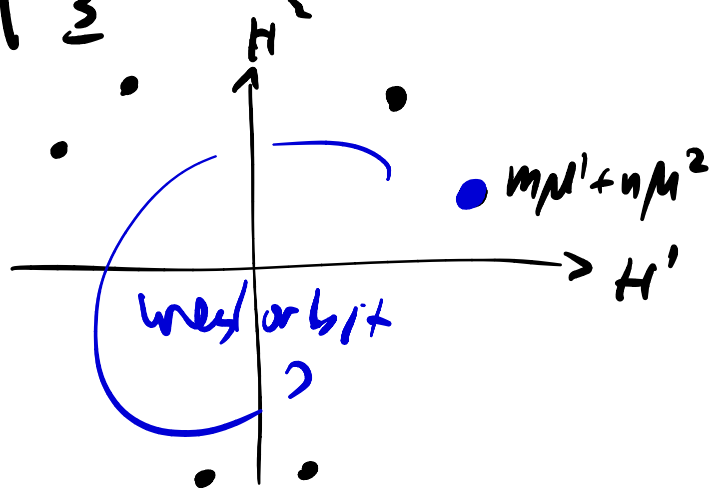
$\overline{3} = (0\ 1) = \begin{matrix}\square\\\square\end{matrix}$

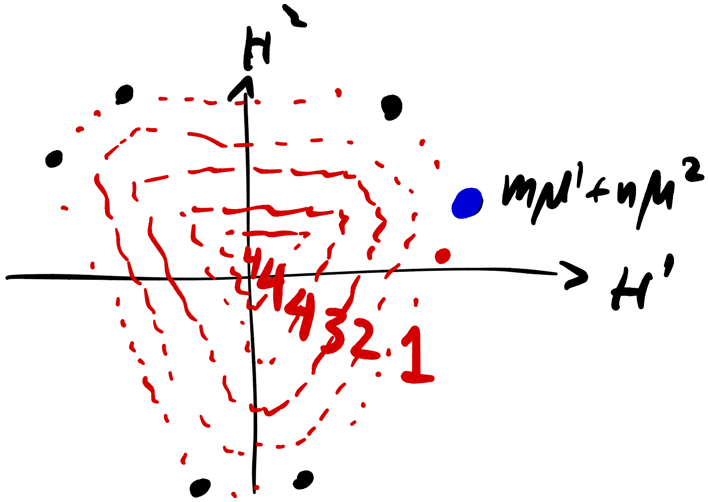
$\overline{6} = (0,2) = \begin{matrix}\square\square\\\square\square\end{matrix} = \mathrm{Sym}^2 \overline{3}$

$\overline{10} = (3,0) = \square\square\square = \mathrm{Sym}^3 \underline{3}$.

CLAIM:
$\overline{3} = \Lambda^2 \underline{3}$

$R_{(m,n)}$





3.5 : everything from simple roots

Imagine someone hands you $\alpha^1 .. \alpha^r$ (eg Dynkin diagram)

All roots from simple roots

Any positive root is $\phi_k = \sum_\alpha k_\alpha \alpha \quad k_\alpha \geq 0.$

$k \equiv \sum_\alpha k_\alpha$ "compositeness".

Q: which of these are roots?

A: by induction on k. $k=1$ ✓

Suppose we know roots ϕ_k for $k \leq \tilde{k}$.

$E_\alpha |\phi_k\rangle = |[E_\alpha, E_{\phi_k}]\rangle.$

if not zero, $\phi_k + \alpha$ is a root.

As an el't of a irrep of $SU(2)_\alpha$

$|\phi_k\rangle$ has $\frac{2\alpha \cdot \phi_k}{\alpha^2} = \ell - r$

$\Rightarrow$ we know ℓ $\Rightarrow$ we know r.

If $r > 0$ then $\phi_k + \alpha$ is a root ∎

eg: $k=1$. $\phi_1 = \beta$ is a simple root.

$\Rightarrow |\beta\rangle$ has $\ell = 0$

$$\frac{2\alpha\cdot\phi_1}{\alpha^2} = \frac{2\alpha\cdot\beta}{\alpha^2} = \ell - r = -r$$

$r=0$ if $\alpha\cdot\beta = 0$. $\Rightarrow$ $\alpha+\beta$ is not a root

otherwise $\alpha+\beta$ is a root.

Any $\phi_{k+1} = \phi_k + \alpha$.

for $SU(3)$: $\alpha^{1,2} = (1, \pm\sqrt{3})/2$ $A_{ab} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}_{ab}$

$A_{12} = -1 \Rightarrow r=1$ for $|\alpha^2\rangle$ wrt $SU(2)_{\alpha^1}$

$A_{21} = -1 \Rightarrow r=1$ $|\alpha^1\rangle$ .. $SU(2)_{\alpha^2}$

$$\Rightarrow E_{\alpha^1}|\alpha^2\rangle \propto |\alpha^1+\alpha^2\rangle$$

$$E_{\alpha^2}|\alpha^1\rangle \propto |\alpha^1+\alpha^2\rangle.$$

(unique generator for each nonzero root $\Rightarrow$ same state.)

$|\alpha^1+\alpha^2\rangle$ has $\ell=1$ for $SU(2)_{\alpha^1}$

$$\ell - r = A_{11} + A_{12} = 2-2 = 1.$$

$\Rightarrow$ $2\alpha^1+\alpha^2$ is not a root $\Rightarrow r=0$

whole algebra from simple roots

$$[H_i, E_\alpha] = \alpha_i E_\alpha.$$

$$[E_\alpha, E_\beta] = \underset{?}{\underline{\underline{N_{\alpha,\beta}}}} E_{\alpha+\beta}$$

$|\beta\rangle = |E_\beta\rangle$ $\quad$ $l - r = 2\frac{\alpha\cdot\beta}{\alpha^2}$, $l + r = 2j$.

wrt $SU(2)_\alpha$

$$J^3_\alpha |\beta\rangle = \frac{\alpha\cdot\beta}{\alpha^2}|\beta\rangle$$

$$\Rightarrow |\beta\rangle = \eta\, |j, \frac{\alpha\cdot\beta}{\alpha^2}\rangle.$$

In $SU(3)$ what's $[E_{\alpha_1}, E_{\alpha_2}] = ?$

① $J^+_{\alpha_1} |E_{\alpha^2}\rangle = \frac{E_{\alpha^1}}{|\alpha^1|}|E_{\alpha^2}\rangle = E_{\alpha^1}|\alpha^2\rangle$

$$= |[E_{\alpha^1}, E_{\alpha^2}]\rangle.$$

② $|\alpha^2\rangle$ has $l = 0$, $r = 1$ wrt $SU(2)_{\alpha^1}$.

$$A_{12} = -1 = l - r.$$

$$\Rightarrow j = (l+r)/2 = 1/2, \quad m = (l-r)/2 = -1/2.$$

$$J^+_{\alpha^1}|E_{\alpha^2}\rangle = J^+_{\alpha^1}|\tfrac{1}{2}, -\tfrac{1}{2}\rangle \overset{SU(2)}{=} \frac{1}{\sqrt{2}}|\tfrac{1}{2}, \tfrac{1}{2}\rangle$$

$$= \frac{\eta}{\sqrt{2}} |E_{\alpha^1+\alpha^1}\rangle$$

$$\Rightarrow |E_{\alpha^1+\alpha^2}\rangle = \sqrt{2}\,\eta\, |[E_{\alpha^1}, E_{\alpha^2}]\rangle$$

$$\Leftrightarrow \quad E_{\underbrace{\alpha^1+\alpha^2}_{=\alpha^3}} = \sqrt{2}\,\eta\, [E_{\alpha^1}, E_{\alpha^1}].$$

η a phase is arbitrary.

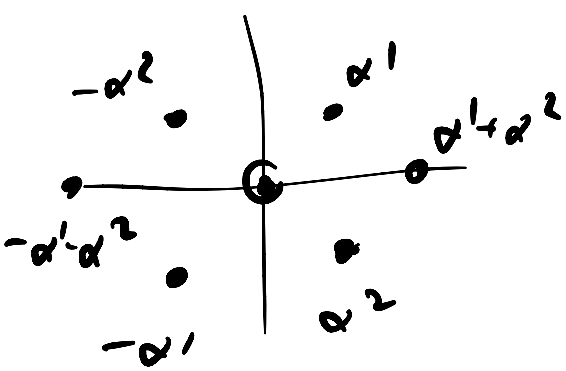
$$|\phi_k\rangle = |\sum_b k_b \alpha^b\rangle \quad \text{w/ } k = \sum_b k_b .$$

raise w/ E_{α^b} $\qquad k_b \to k_b + 1, \; k \to k+1$

$$\ell^a - r^a \to \ell^a - r^a + A_{ab}.$$

$$[E_{-\alpha^1}, E_{\alpha^1+\alpha^2}] = \sqrt{2}\,\eta\, [E_{-\alpha^1}, [E_{\alpha^1}, E_{\alpha^2}]]$$

$$= -[E_{\alpha^1}, \cancel{[E_{\alpha^2}, E_{-\alpha^1}]}_{0}] - [E_{\alpha^2}, \underbrace{[E_{-\alpha^1}, E_{\alpha^1}]}_{-\alpha^1\cdot H}]$$



$$= A\, \alpha^1\cdot\alpha^2\, E_{\alpha^2}$$

$$= \frac{\eta}{\sqrt{2}} E_{\alpha^2}.$$

Dynkin diagrams

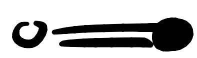
: simple root $\to$ circle

if $\alpha\cdot\beta = 0$ no line

else $\dfrac{\alpha^2}{\beta^2} = \begin{matrix}1\\2\\3\end{matrix}$ draw $\begin{matrix}1\\2\\3\end{matrix}$ lines

circle for shorter roots gets filled in.

eg: $SU(2)$  $SU(3)$ 

$SO(5)$  G_2 

eg G_2 : $A_{ab} = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}_{ab} = 2\dfrac{\alpha_a\cdot\alpha_b}{(\alpha^a)^2}$

$\alpha^1 = (0,1)$ $\alpha^2 = (\sqrt{3}, -3)/2$. ✓

$\cos\theta_{\alpha^1\alpha^2} = -\frac{\sqrt{3}}{2}$ $\theta_{\alpha^1\alpha^2} = 150°$.

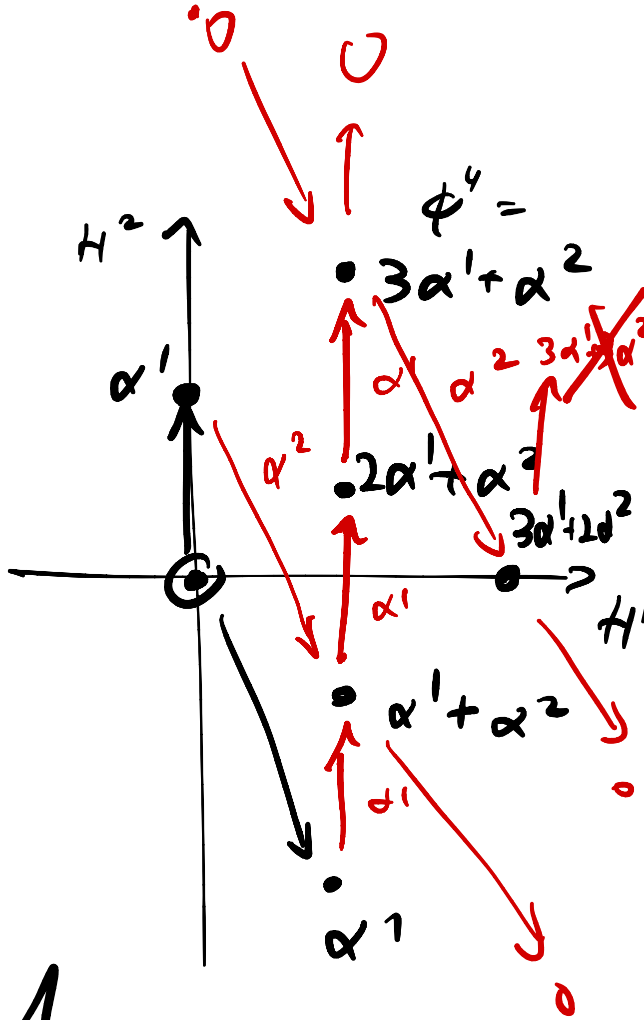
$|\alpha^2\rangle$ has $\dfrac{2\alpha^1\cdot\alpha^2}{(\alpha^1)^2} = -3 = \cancel{\ell^1} - r^1 = \underline{\underline{-r^1}}$ wrt $SU(2)_{\alpha^1}$ can be raised $3\times$ by α^1

$|\alpha^1\rangle$ has $\dfrac{2\alpha^2\cdot\alpha^1}{(\alpha^2)^2} = -1 = \ell^2 - r^2 = -r^2$ wrt $SU(2)_{\alpha^2}$ can be raised once by α^2.

So far: $\phi_2 = \alpha^1+\alpha^2$

$\phi_3 = 2\alpha^1+\alpha^2$

$\phi_4 = 3\alpha^1+\alpha^2$.



$\alpha^1+2\alpha^2$ ✗

$4\alpha^1+\alpha^2$ ✗

$2\alpha^1+2\alpha^2$ ✗

$=2(\alpha^1+\alpha^2)$.

$\phi_5 \overset{?}{=} 3\alpha^1+2\alpha^2$

$|\phi_4\rangle$ has wt $SU(2)_{\alpha^2}$

$$\cancel{l^2} - r^2 = \frac{2\alpha^2\cdot\phi_4}{(\alpha^2)^2} = 3A_{12}+A_{22}$$

$$= -3+2 = -1 \quad ✓$$

$\Rightarrow \underline{\underline{r^2 = +1}} \Rightarrow 3\alpha^1+2\alpha^2$ is a root.

$$\Rightarrow \dim G_2 = 2+6-6 = 14.$$

Next step: fund. wts $\frac{2\alpha^a\cdot\mu^b}{(\alpha^a)^2} = \delta^{ab}$.

construct $\underline{R_{n\mu^1+m\mu^2}}$. R_{μ^1} = fundamental rep.

claim: 1) $\mathrm{Sym}^2 R_{\mu'} \ni \underline{1}$.

$\Rightarrow G_2 \subset SO(\underbrace{\dim R_{\mu'}}_{=d})$

2) $\Lambda^3 R_{\mu'} \ni \underline{1}$.

3.6. Classification of simple Lie algebras.

The simple roots $\{\alpha\}$ of a *simple* Lie alg $\mathfrak{g}$ satisfy:

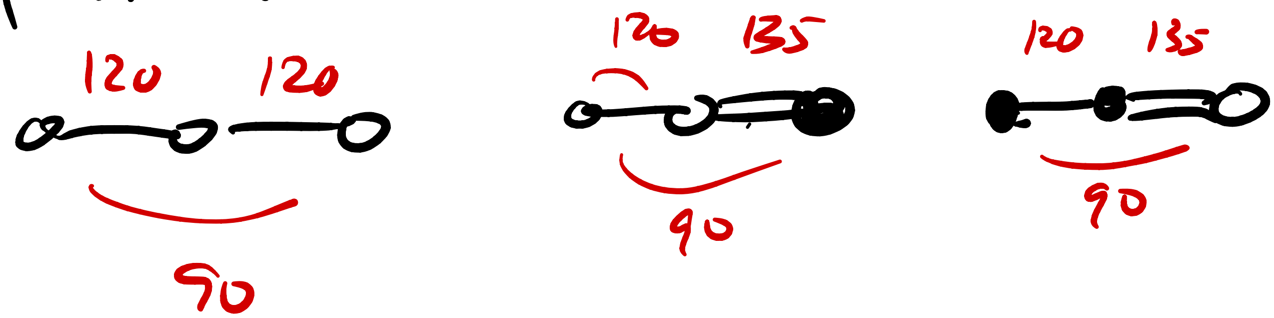
A) They are $r = \mathrm{rank}(\mathfrak{g})$ linearly indep vectors.

B) $A_{\alpha\beta} = \dfrac{2\alpha\cdot\beta}{\alpha^2} \in \{0, -1, -2, -3\}$ $(\alpha \neq \beta)$.

C) They are indecomposable $\Leftrightarrow$ simplicity

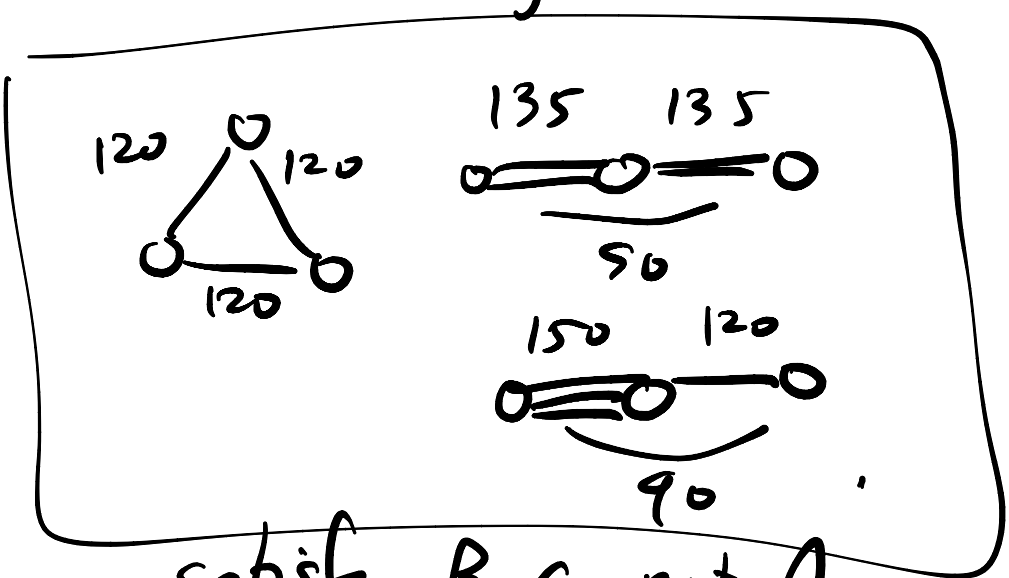
i.e. Dynkin diagram is connected.

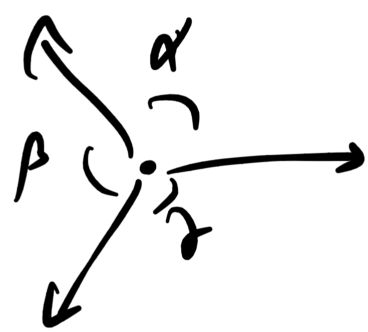
Lemma 1: At rank 3 the only possibilities are:



Pf: for 3 linearly indep vectors

$$\sum_{i<j} \theta_{ij} < 2\pi .$$





$\alpha+\beta+\gamma = 2\pi$.

satisfy B, C not A. ∎