last time: claim: $e^{i\alpha_A X_A} e^{i\beta_B X_B} = e^{i\gamma_C X_C}$

$A, B = 1..\dim G$

if $[X_A, X_B] = i f_{ABC} X_C$

To show: $e^{A+B} = e^A G$

G is a function only of $\mathrm{ad}_A^n(B)$

$\mathrm{ad}_A(\cdot) \equiv [A, \cdot]$. $\mathrm{ad}_A^2(\cdot) = [A,[A,\cdot]]$ ...

Pf: $G(s) \equiv e^{-sA} e^{s(A+B)}$

$$\partial_s G(s) = -AG(s) + e^{-sA}(A+B)\, e^{s(A+B)}$$

$e^{sA}e^{-sA}$

$$= B(s) G(s)$$

$$B(s) \equiv e^{-sA} B e^{sA} = e^{-s\,\mathrm{ad}_A} B$$

(Recall: interaction-picture time evolution) $\left(e^{\mathrm{ad}_A} = \mathrm{Ad}_{e^A}\right)$

$$G(s) = G(0) + \int_0^s dt\, B(t)\, G(t)$$

$$= G(0) + \int_0^s dt_1\, B(t_1)\left(G(0) + \int_0^{t_1} dt_2\, B(t_2) G(t_2)\right)$$

$$G(s) = \sum_{n=0}^{\infty} \int_0^s dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \cdots \int_0^{t_n} dt_{n-1} \; B(t_1)\, B(t_2) \cdots B(t_n)$$

$$T\, B(t_1)\, B(t_2) \equiv \begin{cases} B(t_1)\, B(t_2) & \text{if } t_1 \geq t_2 \\ B(t_2)\, B(t_1) & \text{if } t_2 > t_1 \end{cases}$$

$$G(s) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^s dt_1 \cdots \int_0^s dt_n \; T\, B(t_1) \cdots B(t_n)$$

$$= T\left(e^{\int_0^s dt\, B(t)}\right)$$

$G(0) = 1$

$$e^{-A} e^{A+B} = G(1) = T\, e^{\int_0^s dt\, e^{-t\, ad_A} B}$$

$$ad_A B = \sum \gamma_A X_A \,.$$

BCH formula: $e^X e^Y = e^{(?)}$

$$\log\left(e^X e^Y\right) = X + \int_0^1 dt \underbrace{g\left(e^{ad_X} e^{t\, ad_Y}\right)}_{\text{linear operator defined by series exp of } g(z) \text{ about } z=1.}(Y)$$

$$g(z) \equiv \frac{\log z}{1 - 1/z} \,.$$

Pf ingredients: ① $\partial_t e^{X+tY}\Big|_{t=0} = \int_0^1 ds\, e^{(1-s)X} Y e^{sX}$

(product rule) $= e^X \dfrac{1-e^{-ad_X}}{ad_X} Y$

hint: $e^{X+tY} = \left(e^{\frac{X}{n}+\frac{tY}{n}}\right)^n \quad \forall n.$

② (chain rule) $e^{-Z(t)}\partial_t e^{Z(t)} = \dfrac{1-e^{-ad_{Z(t)}}}{ad_{Z(t)}} \partial_t Z(t)$

③ let $Z(t) \equiv e^X e^{tY}$ $\quad e^{Z(t)} = e^X e^{tY}$

$e^{-Z(t)}\partial_t e^{Z(t)} = (e^X e^{tY})^{-1} e^X e^{tY} Y = Y.$

④ solve for $\partial_t Z(t) = \underbrace{\left(\dfrac{1-e^{-ad_{Z(t)}}}{ad_{Z(t)}}\right)^{-1}}_{g(e^{ad_{Z(t)}})} Y$

$g(z) = \left(\dfrac{1-z^{-1}}{\log(z)}\right)^{-1}$

⑤ $\int_0^1 dt(\text{RHS})$

$\boxed{ad_X^n Y = 0 \quad n > \ldots}$ most ~~useful~~ when

3.1 Lie algebras & structure constants

$$[X_A, X_B] = i f_{ABC} X_C$$

f_{ABC} ← structure const

- $[A,B] = -[B,A] \Rightarrow f_{ABC} = -f_{BAC}$

- for unitary reps $\Rightarrow X_A = X_A^\dagger$

$$[X_A, X_B]^\dagger = -i f^*_{ABC} X_C$$
$$= -i f_{ABC} X_C$$

so $\underline{f_{ABC} = f^*_{ABC}}$ is real

- Jacobi identity

$$0 = [X_A,[X_B,X_C]] + [X_B,[X_C,X_A]] + [X_C,[X_A,X_B]]$$

OR $f_{BCD} f_{ADE} + f_{ABD} f_{CDE} + f_{CAD} f_{BDE} = 0.$

OR

$$\boxed{[\mathrm{ad}_X, \mathrm{ad}_Y] = \mathrm{ad}_{[X,Y]}}.$$

Adjoint Rep of $\mathfrak{g}$ (any Lie alg):

$$(T_A) \qquad (T_A)_{BC} \equiv -i f_{ABC}.$$

$$\xrightarrow{\text{Jacobi}} \quad [T_A, T_B] = i f_{ABC} T_C.$$

Jacobi id says $X \to ad_X$ is a rep of $\mathfrak{g}$.

$\dim R_{adj} = d_G$. dim of group.

$$V_{adj} = \text{span} \{ | X_A \rangle \} \quad A = 1..d_G$$

$$D_{adj}(X_A) | X_B \rangle = | [X_A, X_B] \rangle$$

$$= | i f_{ABC} X_C \rangle$$

$$= i f_{ABC} | X_C \rangle$$

$\boxed{\text{sign?}}$ $$= (T_A)_B{}^C | X_C \rangle.$$

For any Rep: ~~aaaa~~ an inner product

& $\mathrm{tr}_R X^A X^B \equiv K^{AB}$ symmetric.

By $(X^A)' = L_{AB} X^B$ diagonalize K^{AB}.

$\mathrm{tr}\, X'^A X'^B = K^A \delta_{AB}$

rescale $\underline{X'^A = \lambda k^A X''^A}$.

Can set $|K^A| = \lambda$.

<u>claim</u>: for compact Lie groups $k^A > 0$.

$\left(\Rightarrow \subset SO(n) \quad \exists s_* \ \underline{e^{i s_* X} = 1} \right)$

If $\underline{\mathrm{tr}\, X^A X^B = \lambda \delta^{AB}}$

$\mathrm{tr}((\text{Lie alg})\, X^D) \Longrightarrow$

$$f_{ABC} = -\frac{i}{\lambda} \mathrm{tr}[X^A, X^B] X_C$$

$$\underset{\uparrow}{=} f_{BCA} = -f_{BAC} = -f_{ACB}$$
$$= -f_{CBA}.$$

cycl. of tr, IBP.

In such a basis T_A are imaginary + antisymmetric

$\Rightarrow$ hermitian

$\Rightarrow R_{adj}$ is unitary.

$\langle X^A | X^B \rangle = \mathrm{tr}\, T^A_{adj} T^B_{adj}$.

<u>Def</u>: If $T^R_A = T^{R\dagger}_A$ then R is a unitary rep of $\mathfrak{g}$.

$\Longleftrightarrow$ $e^{i\alpha\cdot T^R}$ is unitary

$\rightarrow$ R is unitary as a rep of G.

<u>examples</u>: $SO(n)$.

$O(n) = \{ n\times n$ real matrices w/ $R^T R = \mathbb{1}\}$

$(O(n))_0 =$ component connected to $\mathbb{1}$

$= SO(n) = \left\{ \begin{array}{l} R \in O(n) \\ \text{w/ } \det R = \mathbb{1} \end{array} \right\}$

$$R(\theta) = \mathbb{1} + A + \dots$$

$$\Rightarrow \mathbb{1} = R^T R = (\mathbb{1} + A^T)(\mathbb{1} + A) + \dots$$

$$= \mathbb{1} + A^T + A + \dots$$

$\Longleftrightarrow$ $\underline{A^T = -A}$ is <u>A.S.</u>

$\boxed{n=2}$ only one 2×2 AS matrix:

$$\mathcal{J} \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

so $A = \theta \mathcal{J}$

$$R = \mathbb{1} + \theta\mathcal{J} + O(\theta^2) = \begin{pmatrix} 1 & \theta \\ -\theta & 1 \end{pmatrix} + O(\theta^2)$$

$$R(\theta) \overset{\text{Lie}}{=} \lim_{N\to\infty} \left(1 + \frac{\theta\mathcal{J}}{N}\right)^N = e^{\theta\mathcal{J}}$$

$$\overset{\text{Taylor}}{=} 1 + \theta\mathcal{J} + \frac{\theta^2}{2}\mathcal{J}^2 + \dots = \sum_{k=0}^{\infty} \frac{\theta^k \mathcal{J}^k}{k!}$$

$$\mathcal{J}^2 = -\mathbb{1} \Rightarrow \begin{cases} \mathcal{J}^{2\ell} = \mathbb{1}(-1)^\ell \\ \mathcal{J}^{2\ell+1} = (-1)^\ell \mathcal{J} \end{cases}$$

$$R(\theta) = \cos\theta\, \mathbb{1} + \sin\theta\, \mathcal{J} = \begin{pmatrix} \cos\theta & +\sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$\begin{cases} J \equiv -i\mathcal{J} \\ = J^\dagger . \end{cases} \qquad = e^{-i\theta J}$$

$\boxed{n=3}$ $A = \theta^A \mathcal{J}^A$

$$\mathcal{J}^1 = \begin{pmatrix} 0 & & \\ & & 1 \\ & -1 & \end{pmatrix} \quad \mathcal{J}^2 = \begin{pmatrix} & & 1 \\ & 0 & \\ -1 & & \end{pmatrix} \quad \mathcal{J}^3 = \begin{pmatrix} & 1 & \\ -1 & & \\ & & 0 \end{pmatrix}$$

package: $(\mathcal{J}_i)^j{}_k = \epsilon_{ijk}$

$$\epsilon_{ijk} = -\epsilon_{jik} \dots = \epsilon_{jki} \qquad \epsilon_{123} = 1$$

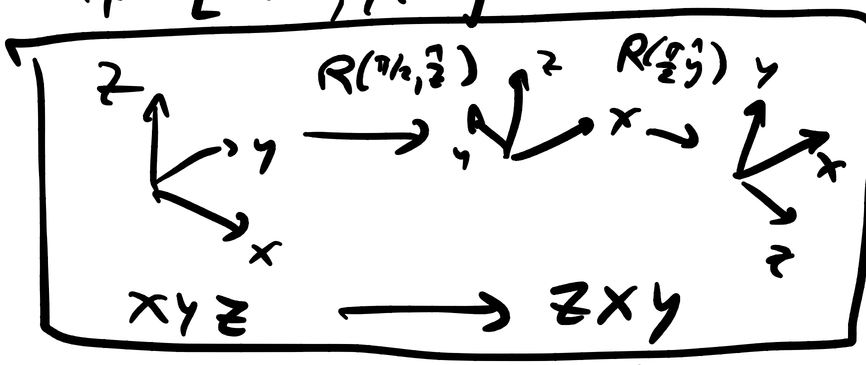
$$J^A = -i\,\mathcal{J}^A .$$

$$R(\theta^A) = e^{\theta^A \mathcal{J}^A} = e^{i\theta^A J^A} = e^{i\theta\,\hat{n}\cdot\tilde{J}} = R(\hat{n},\theta).$$

$$R = \mathbb{1} + A + \dots$$

$$R R' R^{-1} = (\mathbb{1}+A) R' (\mathbb{1}-A) + \dots = R' + [A, R'] + \dots$$

$\underline{R' = \mathbb{1} + A'}$. $\quad = \mathbb{1} + A' + [A, A'] + \dots$



$\underline{n=3}$: $A = \theta\cdot\mathcal{J}$, $A' = \theta'\cdot\mathcal{J}$

$([A, A'])^T = -[A, A']$ & ALSO an AS 3x3 matrix

$$[A, A'] = \theta''\mathcal{J}$$

$$[J_1, J_2] = iJ_3 = -[J_2, J_1] + \text{cyclic}$$

$$[J_i, J_j] = i\epsilon_{ijk}J_k . \qquad su(2) = so(3).$$

$$[\mathcal{J}_i, \mathcal{J}_j] = \epsilon_{ijk} \mathcal{J}_k$$

real AS, real AS, real AS

These are the representation matrices for the 3 of SO(3). But the Lie alg. is the same for my rep.

of SO(n)

something that transforms in the $\underline{n}$ r is $\underline{\text{vector}}$.

$\boxed{\text{general } n}$ $$A = \sum_{m<n} A_{mn} \mathcal{J}_{(mn)}$$

$$\mathcal{J}_{(mn)}^{ij} \equiv \begin{cases} 0 & \text{everywhere except} \\ 1 & \text{in the } mn \text{ entry} \\ -1 & \text{'' '' } nm \text{ ''} \end{cases} \quad \text{so it's AS.}$$

generates rot. in the mn plane $= \delta^{mi}\delta^{nj} - \delta^{mj}\delta^{ni}$

There are $\frac{n(n-1)}{2} = \dim SO(n)$.

$$J_{(mn)} \equiv -i\, \mathcal{J}_{(mn)}.$$

$$R(\theta) = \exp\left(i \sum_{m<n} \theta_{(mn)} J_{(mn)}\right).$$

$n=4$. $[J_{(12)}, J_{(34)}] = 0$.

# indices in common	$[J_{(mn)}, J_{(pq)}]$
0	0
1	erase shared index
2	0

$$\star\ [\underbrace{J_{(12)}}_{J_z}, \underbrace{J_{(23)}}_{J_x}] = \underbrace{+iJ_{(13)}}_{+iJ_y} = -i\, J_{(31)}$$

$$[J_{(mn)}, J_{(mq)}] = i\, J_{(nq)}$$

erase shared index.

Generally:

$$\Big[\, [J_{(mn)}, J_{(pq)}] = i\big(\delta_{mq} J_{(nq)} + \dots\ \big) \Big]_{SO(4)}$$

AS under $m \leftrightarrow n$, $p \leftrightarrow q$

$(mn) \leftrightarrow (pq)$

$SO(4)$ let $J_i = \epsilon_{ijk} J_{(jk)}$, $K_i = J_{(i4)}$

$i = 1..3$

$$[J_i, J_j] = i\,\epsilon_{ijk} J_k$$

$$[K_i, K_j) = i\,\epsilon_{ijk} J_k$$

$$[J_i, K_j] = i\,\epsilon_{ijk} K_k$$

$SO(3) \subset SO(4)$

$(\vec{x}, x_4) \to (\vec{x}, -x_4)$

$\vec{J} \to \vec{J}$, $K \to -K$

"K is a vector of SO(3)"

$$K^j \xrightarrow{SO(3)} e^{i\theta^i \mathrm{ad}_{J^i}} K^j = K^j + i\theta^i [J^i, K^j] + O(\theta^2)$$

$$e^{i\,\mathrm{ad}_{\theta J}} = \mathrm{Ad}_{e^{i\theta J}} \qquad = R(\theta) K^j R(\theta)^{-1}$$

eg $\quad e^{-i\varphi J^3} K_1 e^{+i\varphi J^3} = \cos\varphi K_1 + \sin\varphi K_2$

$\vec{J}_\pm = \frac{1}{2}(\vec{J} \pm \vec{K})$ $\qquad$ claim: each satisfies $SO(3) = SU(2)$

and $[J_\pm^i, J_\mp^j] = 0.$

$SO(4) = SU(2) \times SU(2)$

A similar story holds for $SO(p,q)$

(eg $p=3,\ q=1.$)

$R^T R = \mathbb{1} \rightsquigarrow \eta_{\mu\nu} = \Lambda_\mu{}^\nu \eta_{\nu\rho} (\Lambda^T)^\rho{}_\sigma$

$= (1+A)_\mu{}^\nu \eta_{\nu\rho} (\mathbb{1}+A^T)^\rho{}_\sigma$

$A_\mu{}^\nu \eta + \eta_{\mu\rho} (A^T)^\rho{}_\sigma = 0.$

if $\mu\nu = i0$ $\quad A_{\mu\nu}$ is symmetric $\xRightarrow{\text{Hermiticity}}$ Real BOOSTS.

if $\mu\nu = ij$ $\quad A_{\mu\nu}$ is AS $\xRightarrow{\text{Hermiticity}}$ Imaginary ROTS

Finite BOOST $\quad e^{\eta B} \quad \eta \in \mathbb{R}.$ "rapidity"

$\Rightarrow$ NONCOMPACT.

Simplicity & semisimplicity: An invariant subalg $\mathfrak{h} < \mathfrak{g}$

$$\mathfrak{h} = \mathrm{span}\{ X \in \mathfrak{g} \mid [X, Y] \in \mathfrak{h} \ \forall\, Y \in \mathfrak{g} \}$$

NORMAL

$\Rightarrow e^{\mathfrak{h}} = H_0 < G_0$ is a NORMAL subgroup.

If $h = e^{iX}$, $g = e^{iY}$ $\Rightarrow \underline{g^{-1} h g = e^{iX'}} \in H$

$X \in \mathfrak{h}$ $\qquad Y \in \mathfrak{g}$

$$\mathfrak{h} \ni X' = e^{-iY} X e^{iY} = e^{-i\,\mathrm{ad}_Y} X$$

$$= \sum_k \frac{(-i)^k}{k!} \underbrace{\mathrm{ad}_Y^k (X)}_{\in \mathfrak{h}}$$

$0, \mathfrak{g}$ are trivial inv't subalgebras.

If $\mathfrak{g}$ has no nontrivial subalgebras, $\mathfrak{g}$ is simple.

$(\Rightarrow e^{\mathfrak{g}}$ is a simple group.)

The adj. of a simple alg is irreducible

else $\exists$ invariant subspace $V = \mathrm{span}\{T_r\}$

$V^\perp = \mathrm{span}\{T_x\}$

$(T_A)_{xr} = 0 = -i f_{Axr} \ \forall A = r, x \Rightarrow f_{xx'x''}, f_{rr'r''}$.

1d invariant subalgebra : $\Rightarrow$ U(1) factor. T_A

$\Rightarrow$ $f_{ABC} = 0 \quad \forall\, BC.$

$\Rightarrow$ $\mathrm{tr}\, T_A T_B$ is degenerate.

A Lie alg w no U(1) factors $\equiv$ semisimple.

$= \bigoplus$ (simple lie alg.)

$\Rightarrow$ every X_A fails to commute w someone

$\Rightarrow$ appears on RHS of some commutator.

Big picture of Rep thy of Lie alg.

tensor methods

$V^i \in \underline{n}$

$T^{ij} \in \underline{n} \otimes \underline{n}$

$T^{ijk} \in \underline{n} \otimes \underline{n} \otimes \underline{n}$

$\rightarrow$ tensor network diagrams.

Cartan-Weyl method

exactly like QM

label states by

evals of a CSCO

$\equiv$ Cartan subalgebra.

$\rightarrow$ classification ...