Induced Reps. construct a rep of G from a rep of $H \subset G$.

$D^W(h): W \to W \quad \rightsquigarrow \mathrm{Ind}_H^G(W)$ has carrier space

$$W \times V_{G/H}$$

$$V_{G/H} = \mathrm{span}\{|x\rangle, x \in G/H\}$$

identity coset

$a_x \in G, \quad g a_x = a_{gx} h$

$$\left\{ \begin{array}{l} D(h)|n,0\rangle = \underline{|m,0\rangle} (D^W(h))_{mn} \\ D(a_x)|n,0\rangle = |m,x\rangle \end{array} \right.$$

$+$ it's a rep. $\Rightarrow D(g)|n,x\rangle = |m,gx\rangle D^W(h)_{mn}$

If $|f\rangle = \sum_{nx} \underline{f_n}(x)|nx\rangle$

$f_n(x) \longmapsto D^W_{mn}(h) f_m(g^{-1}x)$.

claim: different choices of reps $\Longleftrightarrow$ basis transf.

Simplest eg: $H = \mathbb{Z}_2 = \langle \sigma | \sigma^2 = e\rangle \subset G = \mathbb{Z}_4 = \langle \tau | \tau^4 = e\rangle$

$\sigma = \tau^2$

W: $\boxed{\underline{D(\sigma) = -1}}$

$$\mathbb{Z}_4/\mathbb{Z}_2 = \{\underset{(\sigma,\tau^2)}{e}, \underset{(\tau^3)}{\tau}\}.$$

$$\Rightarrow V_{\mathbb{Z}_4/\mathbb{Z}_2} = \mathrm{span}\{|e\rangle, |\tau\rangle\}.$$

$D(\tau)|e\rangle = |\tau\rangle$. $D(\tau)|\tau\rangle = D(\tau)D(\tau)|e\rangle = D(\tau^2)|e\rangle = D(\sigma)|e\rangle$

$= \underline{-}|e\rangle$.

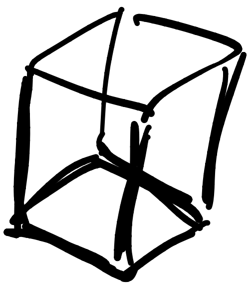
$D(e) = \begin{pmatrix}1 & \\ & 1\end{pmatrix}$ $\quad D(\tau) = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$

$D(\tau^2) = \begin{pmatrix}-1 & \\ & -1\end{pmatrix}$ $\quad D(\tau^3) = D(\tau)D(\tau^2) = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$. ✓

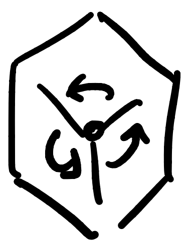
$$\chi_{\mathrm{Ind}_{\mathbb{Z}_2}^{\mathbb{Z}_4}(-1)} = \begin{pmatrix}2\\0\\-2\\0\end{pmatrix} = \begin{pmatrix}1&1&1&1\\1&i&-1&-i\\1&-1&1&-1\\1&-i&-1&i\end{pmatrix}\begin{pmatrix}0\\1\\0\\1\end{pmatrix} \begin{array}{l}\leftarrow 1_0 \leftarrow k\\ \leftarrow 1_1\\ \leftarrow 1_2\\ \leftarrow 1_3\end{array}$$

$$\Longrightarrow \mathrm{Ind}_{\mathbb{Z}_2}^{\mathbb{Z}_4}(-1) = 1_1 \oplus 1_3 .$$

$$\boxed{D_k(g^\ell) = \omega^{k\ell}}$$

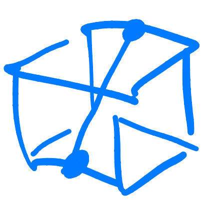


- The $\underline{8}$ of S_4 on vertices is $\mathrm{Ind}_{\mathbb{Z}_3}^{S_4}$ (trivial rep of $\mathbb{Z}_3$)



$\left(|S_4/\mathbb{Z}_3| = 8 . \right)$

$\underline{12}$ rep of edges is $\mathrm{Ind}_{\mathbb{Z}_2}^{S_4}$(trivial rep)



$\mathbb{Z}_2$ fixes 2 edges.

$|S_4| / |H| = \dim R$.

$\pi/4$ fixes 2 faces

$\underline{6}$ on faces $= \mathrm{Ind}_{\mathbb{Z}_4}^{S_4}$(triv)

$\underline{\text{eg}}$: $E(2) =$ transl $+$ $\underset{\supset SO(2)}{\underset{\sim}{\text{rot}}}$ of $\mathbb{R}^2$

Poincaré in 3+1 $=$ trans $+$ $\underset{SO(3)}{\underset{\sim}{\text{rot}}}$ $+$ boosts

little group.

$$\text{Reps of } H \underset{\mathrm{Res}_H^G}{\overset{\mathrm{Ind}_H^G}{\rightleftarrows}} \text{Reps of } G \supset H$$

- If $f(h)$ a function on H $\longrightarrow$ $f: G \to \mathbb{C}$
  $f: H \to \mathbb{C}$ $\qquad g \mapsto \begin{cases} f(g) & \text{if } g \in H \\ 0 & \text{else.} \end{cases}$

make a class function:

$$s \mapsto \mathrm{Ind}_H^G[f](s) = \frac{1}{|H|} \sum_{g \in G} f(g^{-1} s g)$$

$G \to \mathbb{C}$

- Given a rep of G $\quad D^V(g): V \to V$

  $\mathrm{Res}_H^G(V)$ is a rep of H

  w/ $D^V(h): V \to V$ for $h \in H$.

- Given $\phi_{1,2}: G \to \mathbb{C}$

$$\langle \phi_1, \phi_2 \rangle_G = \frac{1}{|G|} \sum_{g \in G} \phi_1(g^{-1}) \phi_2(g).$$

Given ψ, ϕ class f'ns on H, G respectively

claim: $\langle \psi, \mathrm{Res}_H^G[\phi] \rangle_H = \langle \mathrm{Ind}_H^G[\psi], \phi \rangle$

(Res and Ind are adjoints of ea. other.)

an irrep of G : $\operatorname{Res}_H^G V_b = \bigoplus_{\substack{\text{irreps of } H \\ a}} W_a^{\underline{\underline{m_{ab}}}}$

an irrep of H $\quad \operatorname{Ind}_H^G W_a = \bigoplus_{\text{irreps, } b \text{ of } G} V_b^{\underline{\underline{m_{ab}}}}$

$$m_{ab} = \langle \chi_{W_a}, \operatorname{Res}_H^G(\chi_{V_b}) \rangle$$

$$= \langle \operatorname{Ind}_H^G[\chi_{W_a}], \chi_{V_b} \rangle$$

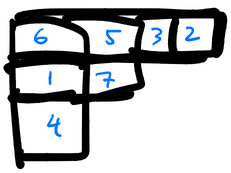
<u>and</u>: $\underline{\chi_{\operatorname{Ind}_H^G[W]}(g) = \operatorname{Ind}_H^G[\chi_W](g)}$.

"frobenius Reciprocity".

Reps of S_n and Young Tableaux:

\# of irreps of G = # of conj. classes

for S_n: ? $\longleftrightarrow$ Young diagrams w/ n boxes



$\longleftrightarrow \left| \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 5 & 3 & 2 & 1 & 7 & 4 \end{pmatrix} \right\rangle$ $\in$ Regular Rep of S_7.

$\equiv |6532174\rangle$

(WARNING: not cycle notation!)

$$D(\pi)\,|i_1 i_2 \dots\rangle = |\pi_{i_1} \pi_{i_2} \cdots\rangle$$

Prescription for a given diagram:

- symmetrize rows $\quad |\square\square\rangle \equiv \dfrac{|12\rangle + |21\rangle}{\sqrt{2}}$
- antisymmetrize cols $\quad \left|\begin{smallmatrix}\square\\ \square\end{smallmatrix}\right\rangle \equiv \dfrac{|12\rangle - |21\rangle}{\sqrt{2}}$.

$$D(\sigma)|\square\square\rangle = +|\square\square\rangle, \quad D(\sigma)\left|\begin{smallmatrix}\square\\ \square\end{smallmatrix}\right\rangle = -\left|\begin{smallmatrix}\square\\ \square\end{smallmatrix}\right\rangle.$$

$$\left|\,\young(12,3)\right\rangle = \big(|123\rangle + |213\rangle - |321\rangle - |231\rangle\big)/\sqrt{4}\,.$$

$$\left|\,\young(32,1)\right\rangle = \big(|321\rangle + |231\rangle - |123\rangle - |213\rangle\big)/\sqrt{4}\,.$$

$$\begin{pmatrix} \left|\young(12,3)\right\rangle \\ \left|\young(32,1)\right\rangle \\ \left|\young(23,1)\right\rangle \\ \left|\young(13,2)\right\rangle \\ \left|\young(31,2)\right\rangle \\ \left|\young(21,3)\right\rangle \end{pmatrix} = \begin{array}{c} (13) \\ \\ (21) \\ \\ (23) \\ \\ \end{array} \begin{pmatrix} 1 & 1 & -1 & -1 & 0 & 0 \\ -1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & -1 \\ 0 & 0 & -1 & -1 & 1 & 1 \\ -1 & -1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} |123\rangle \\ |213\rangle \\ |321\rangle \\ |231\rangle \\ |132\rangle \\ |312\rangle \end{pmatrix}$$

$\underset{(13)}{1+2=0}$, $\underset{(21)}{3+4=0}$ $\underset{(23)}{5+6=0}$ $\underset{(123)}{1+3+5=0}$, $\underset{(321)}{2+4+6=0}$

The 2-dim'l subspace, is spanned by $\underline{\young(\cdot\cdot,\cdot)}$

where #s are meaning $\left(\young(12,3)\,,\ \young(13,2)\right)$.

S_2 : $\begin{ytableau} 1 & 2 \end{ytableau} \oplus \begin{ytableau} 1 \\ 2 \end{ytableau} = 1 \oplus 1'$

S_3 : $\begin{ytableau} 1 & 2 & 3 \end{ytableau} \oplus \mathrm{span}\left(\begin{ytableau} 1 & 2 \\ 3 \end{ytableau} , \begin{ytableau} 1 & 3 \\ 2 \end{ytableau} \right) \oplus \begin{ytableau} 1 \\ 2 \\ 3 \end{ytableau}$

$= \underline{1} \oplus \underline{2} \oplus \underline{1}'$

S_4 : $\begin{ytableau} \ & \ & \ & \ \end{ytableau} \oplus \left(\begin{ytableau} 1 & 2 & 3 \\ 4 \end{ytableau} , \begin{ytableau} 1 & 3 & 4 \\ 2 \end{ytableau} , \begin{ytableau} 1 & 2 & 4 \\ 3 \end{ytableau} \right) \oplus$

$\underline{1} \oplus \underline{3} \oplus$

$\left(\begin{ytableau} 1 & 2 \\ 3 & 4 \end{ytableau} , \begin{ytableau} 1 & 3 \\ 2 & 4 \end{ytableau} \right) \oplus \left(\begin{ytableau} 1 & 2 \\ 3 \\ 4 \end{ytableau} , \begin{ytableau} 1 & 3 \\ 2 \\ 4 \end{ytableau} , \begin{ytableau} 1 & 4 \\ 2 \\ 3 \end{ytableau} \right) \oplus \begin{ytableau} \ \\ \ \\ \ \\ \ \end{ytableau}$

$\underline{2} \oplus \underline{3}' \oplus \underline{1}'$

sgn

these are called <u>Young tableaux</u>.

<u>Q:</u> $\dim R_\lambda$ = # of Young tableaux for the diagram λ.

<u>hook rule:</u> $\begin{ytableau} 5 & 2 & 1 \\ 2 \\ 1 \end{ytableau}$ $\quad \begin{ytableau} 4 & 1 \\ 2 \\ 1 \end{ytableau}$ $\quad \begin{ytableau} 3 & 2 \\ 2 & 1 \end{ytableau}$

hook = # of boxes below or to the right of the box including itself

$$\dim R_\lambda = \frac{n!}{\prod_{\text{boxes}} (\text{hook lengths})}$$

$$\dim R_{\begin{smallmatrix} 4 & 1 \\ 2 & \\ 1 & \end{smallmatrix}} = \frac{4!}{4\cdot 1\cdot 2\cdot 1} = 3 \quad \checkmark$$

$$\dim R_{\begin{smallmatrix} 3 & 2 \\ 2 & 1 \end{smallmatrix}} = \frac{4!}{3\cdot 2\cdot 2} = 2 \quad \checkmark$$

<u>Warning</u>: Young diagrams $\longleftrightarrow$ reps of Lie groups $(SU(n),\ SO(n))$

$$\dim R_\lambda^{SO(n) \text{ or } SU(n)} \neq \dim R_\lambda^{S_n}.$$

Projective Reps

$$D(g)\,D(h) = \omega(g,h)\,D(gh)$$

$\omega(g,h) \in U(1)$, i.e. $|\omega(g,h)| = 1$.

$$\omega : G \times G \to U(1)$$

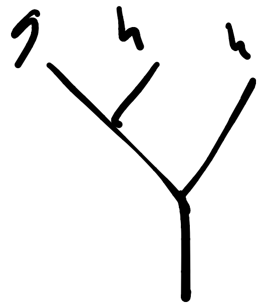
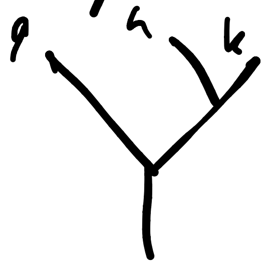
$$D(g)D(h)D(k) = \big(D(g)D(h)\big)D(k) = \omega(g,h)D(gh)D(k)$$

$$= \omega(g,h)\,\omega(gh,k)\,D(ghk)$$

$$= D(g)\big(D(h)D(k)\big)$$

$$= \omega(h,k)\,D(g)\,D(hk)$$

$$= \omega(h,k)\,\omega(g,hk)\,D(ghk)$$

"  $\stackrel{!}{=}$  "

$$1 \stackrel{!}{=} \frac{\omega(g,h)\,\omega(gh,k)}{\omega(h,k)\,\omega(g,hk)}.$$

"2-cocycle".

when are they equivalent? $D(g) \mapsto \gamma(g)D(g)$ $\quad \gamma : G \to U(1)$

$$\gamma(g)D(g)\,\gamma(h)D(h) = \omega(g,h)\,\gamma(gh)\,D(gh)$$

$$\omega(g,h) \sim \omega(g,h)\,\frac{\gamma(gh)}{\gamma(g)\gamma(h)}$$

If $\omega(g,h) = \dfrac{\gamma(gh)}{\gamma(g)\gamma(h)}$ it's a rep!

Let $\Omega^p \equiv \Omega^p(G, U(1)) \equiv$ "p-cochains"

maps : $\underbrace{G \times G \times \cdots G}_{p} \longrightarrow U(1)$

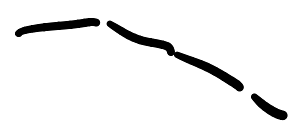
$$\Omega^1 \xrightarrow{\delta_1} \Omega^2 \xrightarrow{\delta_2} \Omega^3$$

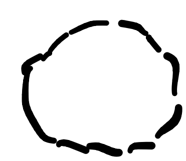
$$\gamma \xmapsto{\delta_1} \delta_1 \gamma(g,h) = \frac{\gamma(h)\gamma(g)}{\gamma(gh)}$$

$$\omega \xmapsto{\delta_2} (\delta_2 \omega)(g,h,k) = \frac{\omega(g,h)\,\omega(gh,k)}{\omega(h,k)\,\omega(g,hk)}$$

$\omega \in \ker \delta_2$ (ie $\delta_2 \omega = 1$)

$\hookrightarrow$ ω is a 2 co<u>cycle</u>.

chain: 

cycle: 

<u>claim:</u> $\delta_2 \circ \delta_1 = 1$

$$\ker \delta_2 \supset \operatorname{Im} \delta_1$$

$$\left\{\begin{array}{c}\text{Inequivalent} \\ \text{projective} \\ \text{reps of } G\end{array}\right\} = \frac{\ker \delta_2 \subset \Omega^2}{\operatorname{Im} \delta_2 \subset \Omega^2} \equiv H^2(G, U(1)).$$

example 1: $G = \mathbb{Z}_2 \times \mathbb{Z}_2 = \langle \sigma, \tau \mid \sigma^2 = e, \tau^2 = e, \sigma\tau = \tau\sigma \rangle$

$D(\sigma) = iX \quad D(\tau) = iY, \quad D(\sigma\tau) = iZ.$

$D(\sigma) D(\tau) = - D(\tau) D(\sigma)$

$\omega(\sigma, \tau) = 1 = -\omega(\tau, \sigma).$

π-rotations in 2 of $SU(2)$.

or: 2 of Q_8.

Any proj. rep of G is an rep of $\tilde{G}$

a central extension of G.

extension of G by an abelian group A:

$$1 \to A \xrightarrow{\psi} \tilde{G} \xrightarrow{\pi} G \to 1$$

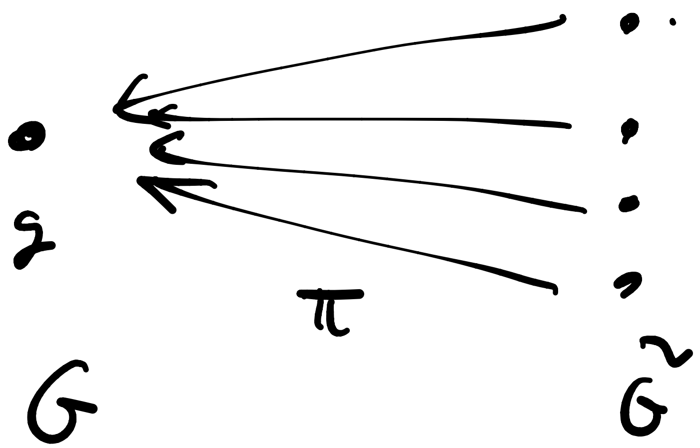
exact seq: $\mathrm{Im}\,\psi = \mathrm{Ker}\,\pi$ (complexes no cohomology)

$\Rightarrow \mathrm{Im}\,\psi \subset \tilde{G}$ is a normal subgp, $\tilde{G}/A$ is a grp. $= G$.

(S_n is an extension of A_n by $\mathbb{Z}_2$)

Central extension means $\operatorname{Im}\psi \subset Z(\tilde{G})$.

$g \in G \xleftarrow{\pi} |A|$ elements of $\tilde{G}$



$|A| = 4$

$$\begin{tikzcd}[row sep=small]
1 \arrow[d] & 1 \arrow[d] & & & & & & & \\
\mathbb{Z}_2 \arrow[d, "\psi"'] & 1 \arrow[d] & -1 \arrow[d] & & & & & & \\
Q_8 \arrow[d, "\pi"'] & 1 \arrow[d] & -1 \arrow[d] & i \arrow[d] & j \arrow[d] & k \arrow[d] & -i \arrow[d] & -j \arrow[d] & -k \arrow[d] \\
\mathbb{Z}_2 \times \mathbb{Z}_2 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] & \tau & \sigma & \tau\sigma & \tau & \sigma & \tau\sigma \\
1 & 1 & 1 & & \cdots & & & &
\end{tikzcd}$$

$\tilde{G}$ is not unique. $D(\sigma) = X \quad D(\tau) = Z$

extends $\mathbb{Z}_2 \times \mathbb{Z}_2$ by $\mathbb{Z}_2$ to get D_4 instead of Q_8.

ex. 2 : $G = U(1) = \{ e^{i\theta}, \theta \in [0, 2\pi) \}$

IRRep of $U(1)$: $U_q(\theta) = e^{iq\theta}$ $q \in \mathbb{Z}$.

eg : $q \in \mathbb{Z} + \frac{1}{2}$ $U(\theta)U(\theta') = U((\theta+\theta')_{2\pi})\,\omega(\theta,\theta')$

$$\omega(\theta,\theta') = \begin{cases} 1 & \text{if } \theta+\theta' \in [0,2\pi) \\ -1 & \text{if } \theta+\theta' \in [2\pi,4\pi) \end{cases}$$

$$1 \to \mathbb{Z}_2 \to U(1) \to U(1) \to 1$$

twice the radius.

ex 3 : $G = SO(3)$ A spinor rep : $\frac{1}{2}$-integer angular momentum

$\underline{2}$: $U(\theta, \hat{n}) = e^{-i\frac{\theta}{2}\hat{n}\cdot\vec{\sigma}} = \cos\theta/2\,\mathbb{1} - i\hat{n}\cdot\vec{\sigma}\sin\theta/2$.

$(\hat{n}\cdot\vec{\sigma})^2 = 1$

is a projective rep of $SO(3)$ $U(2\pi, \hat{n}) = -\mathbb{1}$.

$$U \sigma^a U^\dagger = R_{ab} \sigma^b \qquad R = R(\hat{n}, \theta)$$

in the $\underline{3}$

actual rep

$\underline{\text{of } SO(3)}$.

$\underline{\underline{U \to -U \text{ preserves } R.}}$

$$1 \to \mathbb{Z}_2 \xrightarrow{\begin{pmatrix} -1 & \\ & -1 \end{pmatrix}} SU(2) \xrightarrow{\pi} SO(3) \to 1$$

$$SO(3) = SU(2)/\mathbb{Z}_2 .$$

$$\pi_1(SU(2)) = \pi_1(S^3) = 1 . \implies \underline{\pi_1(SO(3)) = \mathbb{Z}_2}$$

$SO(3) \sim \mathbb{RP}^3$.

$\begin{pmatrix} a & +b \\ -b^* & -a \end{pmatrix}$ w/ det $=$ $|a|^2 + |b|^2 = 1$

$\simeq S^3$.

$\underline{\text{example 4}}$: magnetic $\underline{\text{transl}}$: $G = \mathbb{R}^d$

$$\hat{T}(x) = e^{i p \cdot x / \hbar} = e^{\vec{x} \cdot \vec{\nabla}}$$

$$\hat{T}(x) \psi(\vec{r}) = \psi(\vec{r} + \vec{x}). \quad \text{(Taylor)}$$

$$[p_i, p_j] = \delta_{ij} \longrightarrow \hat{T}(x) \hat{T}(x') = \hat{T}(x + x')$$

$d=3$ $\quad \vec{B} = \vec{\nabla}\times\vec{A}$ uniform.

$$\vec{p} \longrightarrow \vec{\pi} = \vec{p} + \frac{e}{c}\vec{A} \qquad [\pi^i, \pi^j] = -i\frac{\hbar e}{c}\epsilon_{ijk}B^k$$

$$H = \frac{\pi^2}{2m} \quad \text{commutes w/} \quad \tilde{\vec{k}} \equiv \vec{\pi} - \frac{e}{c}\vec{B}\times\vec{r}$$

$$[k^i, k^j] = +i\frac{\hbar e}{c}\epsilon_{jik}B_k$$

$$\hat{T}_B(\vec{x}) \equiv e^{i\tilde{\vec{k}}\cdot\vec{x}/\hbar} \quad \text{satisfy}$$

$$\hat{T}_B(x)\,\hat{T}_B(x') = e^{-i\pi\vec{B}\cdot\vec{x}\times\vec{x}'/\phi_0}\,\hat{T}_B(x+x')$$

$$\phi_0 = \frac{\hbar c}{e}.$$

a proj. rep. of $\mathbb{R}^d$.

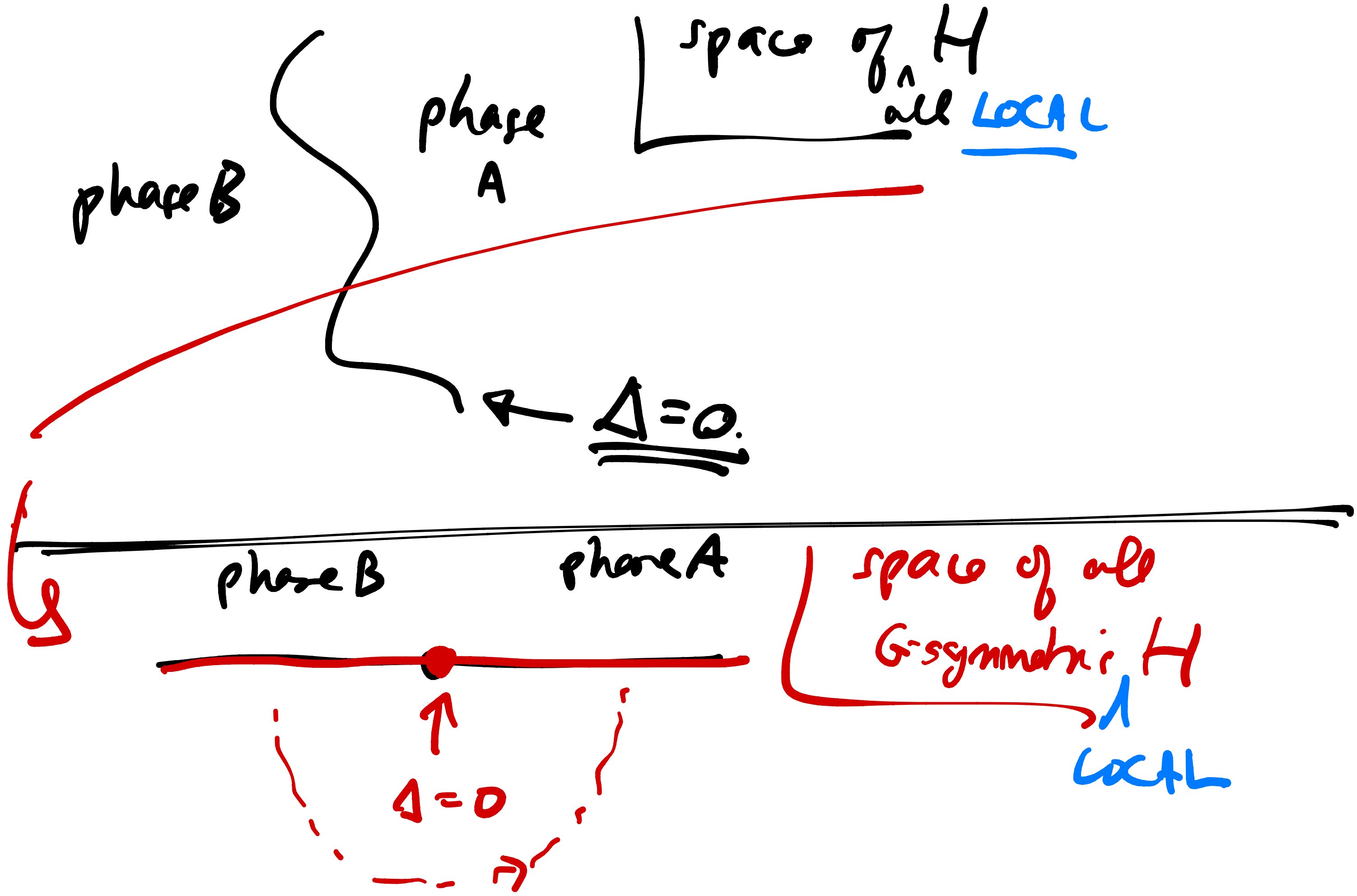
$\boxed{\hbar \to h?}$

Example 5: 1d SPT states "symmetry-protected topological" states.

gapped, symmetric groundstate of a local H w/ G-symmetry.

$\Delta = E_1 - E_0$ is finite $\qquad U|\psi\rangle = |\psi\rangle$

$U = \prod_x u_x$ "on-site". $\qquad \mathcal{H} = \bigotimes_x \mathcal{H}_x$.

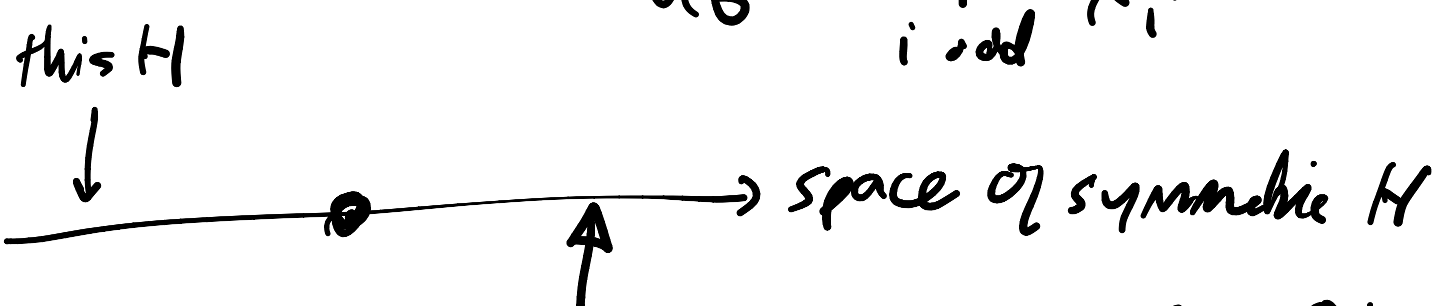


e.g.: $G = \mathbb{Z}_2 \times \mathbb{Z}_2$

$$H = -\sum_i Z_{i-1} X_i Z_{i+1}$$

$$= \langle U_e, U_o \rangle \qquad U_e = \prod_{i \text{ even}} X_i$$

$$U_o = \prod_{i \text{ odd}} X_i$$



$$H_0 = -\sum X_i \qquad |\text{trivial}\rangle = \bigotimes_i |+\rangle_i$$

$$X|+\rangle = |+\rangle.$$