Grand Orthogonality Fact: two irreps $R_{a,b}$ of G:

$$\sum_{g\in G} D^a(g)^*_{ji}\, D^b(g)_{kl} = \delta_{jk}\delta_{il}\delta^{ab}\frac{|G|}{d_a}$$

$\forall\, i\, j\, k\, l\, a\, b$.

Schur's lemma: every irrep of an abelian G is one-dim'l.

2.1 Characters

$$\chi_R : G \longrightarrow \mathbb{C}$$
$$g \longmapsto \chi_R(g)$$

$$\chi_R(g) \equiv \operatorname{tr} D_R(g) = \sum_{i=1}^{\dim R} D_R(g)_{ii}$$

- basis independent

$$\operatorname{tr} S^{-1} D S = \operatorname{tr} D S S^{-1} = \operatorname{tr} D.$$

- class functions

$$\chi(g^{-1}hg) = \operatorname{tr} D(g^{-1}hg) = \operatorname{tr} D(g^{-1}) D(h) D(g)$$
$$= \operatorname{tr} D(h) D(g) D(g)^{-1} = \operatorname{tr} D(h) = \chi(h).$$

- $\chi_{D_1\otimes D_2}(g) = \chi_{D_1}(g)\chi_{D_2}(g)$ $\quad \chi_{D_1\oplus D_2}(g) = \chi_{D_1}(g) + \chi_{D_2}(g)$.

• $\frac{1}{|G|}\sum_{g\in G}\chi_{R_a}(g^{-1})\,\chi_{R_b}(g) = \delta_{ab}$

__UNITARY REP__:

$$\frac{1}{|G|}\sum_{g\in G}\chi_{R_a}(g)^*\,\chi_{R_b}(g) = \delta_{ab}$$

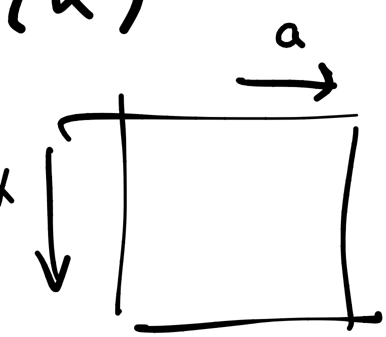
$$\frac{1}{|G|}\sum_{\substack{\text{conj.}\\ \text{classes } \alpha}} n_\alpha\,\chi_{R_a}(\alpha)^*\,\chi_{R_b}(\alpha) = \delta_{ab}$$

($n_\alpha \equiv$ # of el'ts of class α.)

__Inner prod. on class f'ns__:

$$\langle \chi^1, \chi^2\rangle = \frac{1}{|G|}\sum_{\alpha} n_\alpha\,\chi_1(\alpha)^*\,\chi_2(\alpha).$$

"__character table__": $\chi_\alpha{}^a \equiv \chi_{R_a}(\alpha)$



$a = 1 ..$ # of irreps of G

$\alpha = 1 ..$ # of conj. classes of G

__Thm__: Character table is __SQUARE__.

Pf: suppose $\exists$ a class f'n $f(hgh^{-1}) = f(g)$.
not χ of some irrep or sum thereof

$\Rightarrow 0 = \langle f, \chi_a \rangle$.

let $S = \sum_{g \in G} f(g) D^a(g) : V_a \to V_a$

IRREP.

claim: $D^a(h) S = S D^a(h) \quad \forall h \in G$.

Pf: $D^a(h) S$

$= \sum_g f(g) D^a(hg) = \sum_{g' = h^{-1}gh} \underbrace{f(hg'h^{-1})}_{= f(g')} \underbrace{D^a(g'h)}_{D^a(g')D^a(h)}$

$= S D^a(h)$

Schur's lemma $\Rightarrow S = \lambda \mathbb{1}_{V_a}$

tr(BHS) $\Rightarrow \lambda d_a = \operatorname{tr} S = \operatorname{tr} \sum_g f(g) D^a(g)$

$= \sum_g f(g) \operatorname{tr} D^a(g)$

$= \sum_g f(g) \chi^a(g) = |G| \langle f, \chi_{\bar{a}} \rangle^*$

$= 0$.

$D_{\bar{R}^a}(g) = D_{R^a}(g)^{+}$

$\chi_{\bar{R}}(g) = \chi_R(g)^*$

class f'ns on G = $\mathrm{span}_{\mathbb{C}}\{$ characters of irreps, $\chi^a \}$

i.e.

<u>any</u> class f'n

$$F(g) = \sum_{\text{irreps}, a} \chi^a(g) \cdot c_a^F \qquad \text{with}$$

$$c_a^F = \frac{1}{|G|} \sum_{g \in G} \chi_{R_a}(g^{-1}) \, F(g)$$

eg: <u>cyclic groups & Fourier series</u>, $G = \mathbb{Z}_n = \langle g \mid g^n = e \rangle$.

abelian $\Rightarrow$ # of conjugacy classes $= n =$ # of irreps.

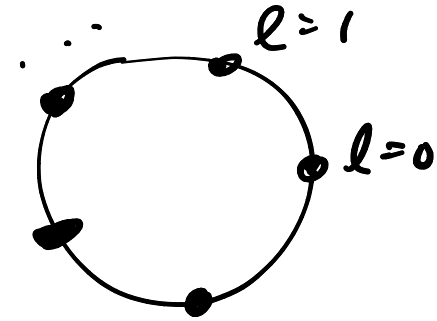
irreps: $D_k(g^l) = \omega^{kl}$ $\qquad k = 0 \ldots n-1$

$l = 0 \ldots n-1$

$\chi_k(g^l) = \omega^{kl}$. $\qquad \omega \equiv e^{2\pi i/n}$.

$$F(l) = \sum_{k=0}^{n-1} \omega^{kl} F_k$$

$$F_k = \frac{1}{n} \sum_{l=0}^{n-1} \omega^{-kl} F(l)$$



$k \in \mathbb{Z}$.

$$\langle \chi_a, \chi_b \rangle = \delta_{ab}$$ "column orthonormality"

also: $$\sum_{\text{irreps},a} \chi_{R_a}(g_\alpha)^* \chi_{R_b}(g_\beta) = \frac{|G|}{n_\alpha} \delta_{\alpha\beta}$$

"row orthonormality".

$$V_{\alpha a} \equiv \sqrt{\frac{n_\alpha}{|G|}} \chi_{R_a}(g_\alpha) \equiv \sqrt{\frac{n_\alpha}{|G|}} \chi^a_\alpha$$

- is square
- ROW orthonormality says $V^\dagger V = \mathbb{1}$

$$(V^\dagger)_{a\alpha} V_{\alpha b} = \delta_{ab}$$

$\Rightarrow$ V is unitary.

[Pf: $V \in GL(n,\mathbb{C})$ is a group
$\Rightarrow$ left inverse = right inverse.]

Any unitary Rep

$$D = \bigoplus_{\text{irreps},a} \underbrace{(D_a \oplus \dots \oplus D_a)}_{m_D^a \text{ times}} \equiv \bigoplus_a D_a^{\oplus m_D^a}$$

$$= \underbrace{D_1 \oplus \dots \oplus D_1}_{m_1 \text{ times}} \oplus \underbrace{D_2 \oplus \dots \oplus D_2}_{m_2 \text{ times}} \oplus \dots$$

$$\chi_D(g) = m_D^1 \chi_1(g) + m_D^2 \chi_2(g) + \dots$$

$$= \sum_{\substack{a \\ \text{irreps}}} m_D{}^a \chi_a(g)$$

$$m_D^a = \langle \chi_a , \chi_D \rangle \equiv \frac{1}{|G|} \sum_{g \in G} \chi_a^*(g) \chi_D(g).$$

<u>eg</u>: R_{reg} : $V_{reg} = \text{span}\{ |g\rangle , g \in G \}$

$$D_{Reg}(g) |h\rangle = \underline{|gh\rangle}$$

is a <u>permutation rep</u>

$\equiv$ also a group action on the basis states.

$$\chi_{Reg}(g) = \text{tr}_{Reg} D_{Reg}(g)$$

$= \#$ of fixed pts of g.

$$= \delta_{e,g} |G|$$

$$\Rightarrow m_{Reg}^a = \frac{1}{|G|} \sum_{g \in G} \chi_a(g)^* \chi_{Reg}(g) = \chi_a(e) = d_a.$$

$$\dim R_{reg} = \boxed{|G| = \sum_{\text{irreps}} d_a^2}.$$

the irrep w/ dim d_a appears d_a times!

eg: $\mathbb{Z}_3 = \langle g \mid g^3 = e \rangle$ $\qquad D_k(g^l) = \omega^{kl}$

$\longrightarrow$ Irreps, q

$\mathbb{Z}_3$	n_c	1	ω	ω^2
e	1	1	1	1
g	1	1	ω	ω^2
g^2	1	1	ω^2	ω

$\downarrow$ classes α

eg: S_3 $\qquad |S_3| = 6 = 1^2 + 1^2 + d^2 \qquad d = \mathrm{tr}_R \mathbb{1}$

$\Rightarrow \underline{d = 2}.$

rep of C_α	n_c	S_3	$\underline{1}$	$\underline{1'}$ sign	$\underline{2}$
$(1) =$	1	·	1	1	2
$(12) =$	3	$\mathbb{Z}_2$	1	-1	$x = 0$
$(123) =$	2	$\mathbb{Z}_3$	1	1	$y = -1$

$r_1 \perp r_2 : 1 - 1 + 2x = 0$

$\Rightarrow x = 0$

$r_1 \perp r_3 : 1 + 1 + 2y = 0$

$\Rightarrow y = -1.$

$1 \cdot 1 \cdot 1 + 3 \cdot 1 \cdot -1 + 2 \cdot 1 \cdot 1$

$= 0$

$\underline{c_1 \perp c_2}$.

$1 \cdot 1^2 + 3 \cdot 1^2 + 2 \cdot 1^2 = 6 = |G|$ ✓

Collider physics of representation thy:

$$R_a \otimes R_b = \bigoplus_{c \text{ irreps}} R_c^{\oplus m_{ab}^c} \qquad m_{ab}^c \in \mathbb{Z}_{\geq 0}$$

$$\chi_{a\otimes b}(g) = \chi_a(g)\chi_b(g) = \sum_c m_{ab}^c \chi_c(g).$$

$$m_{ab}^c = \langle \chi_c, \chi_{a\otimes b} \rangle$$

$$= \frac{1}{|G|} \sum_{g\in G} \chi_c(g)^* \chi_a(g)\chi_b(g).$$

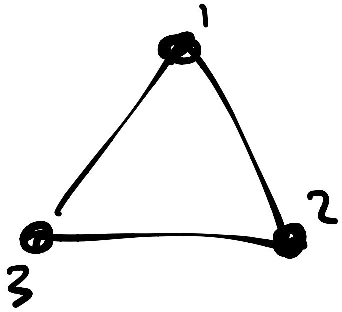
eg for S_3: $a \otimes 1 = a$.

$$\chi_{1'}\chi_{1'} = \chi_1 \qquad \chi_2 \otimes \chi_{1'} = \chi_2$$

$$\chi_2 \otimes \chi_2 = \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix} = \chi_1 + \chi_{1'} + \chi_2$$

$$2 \otimes 2 = 1 \oplus 1' \oplus 2.$$

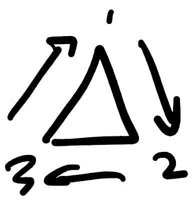
eg 2:



$$\mathcal{H} = \mathrm{span}\{ |j\rangle ,\ j = 1..3 \}$$

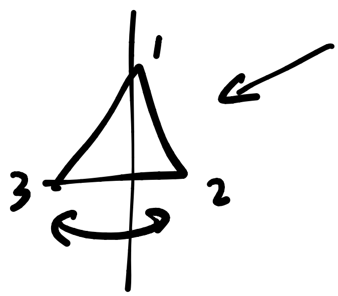
$$H = \sum_{\langle ij\rangle} |i\rangle\langle j| + h.c.$$

Symms of Δ: $(\equiv D_3)$



$\mathbb{Z}_3 \subset D_3$

(123)



$\mathbb{Z}_2 \subset D_3$

(23)

$\underline{D_3 = S_3}$.

$\mathcal{H}$ transforms in $\underline{3}$ of S_3

$$\boxed{D(\pi)|j\rangle = |\pi_j\rangle}$$

$$\chi_3 \begin{pmatrix} (1)(2)(3) \\ (12)(3) \\ (123) \end{pmatrix} \equiv \begin{pmatrix} \chi_3((1)) \\ \chi_3((12)) \\ \chi_3((123)) \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}.$$

[more generally:

$$\chi_{\underline{n} \text{ of } S_n}(\alpha) = \#\text{ of 1-cycles of } \alpha.$$]

$$\alpha\downarrow\begin{pmatrix}3\\1\\0\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix} m_1 + \begin{pmatrix}1\\-1\\1\end{pmatrix} m_{\bar{1}} + \begin{pmatrix}2\\0\\-1\end{pmatrix} m_2$$

$$\cong$$

$$= \alpha\downarrow \overset{\longrightarrow a}{\begin{pmatrix}1 & 1 & 2\\ 1 & -1 & 0\\ 1 & 1 & -1\end{pmatrix}} \begin{pmatrix}m_1\\ m_{\bar{1}}\\ m_2\end{pmatrix}\downarrow a$$

i.e. $\chi^{(3)}_\alpha = \chi_\alpha{}^a \, m_a^{(3)}$

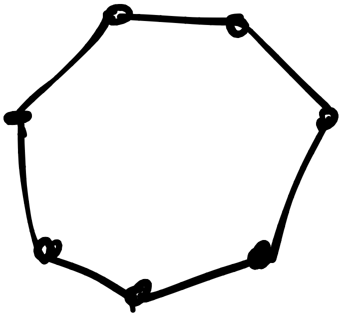
$$\Longleftrightarrow \quad m_a = (\chi^{-1})_a{}^\alpha \, \chi^{(3)}_\alpha$$

$$\chi^{-1}{}_a{}^\alpha = \frac{1}{3!}\, a\downarrow \overset{\longrightarrow \alpha}{\begin{pmatrix}1 & 3 & 2\\ 1 & -3 & 2\\ 2 & 0 & -2\end{pmatrix}}$$

$$m = \begin{pmatrix} \downarrow \end{pmatrix}\begin{pmatrix}3\\1\\0\end{pmatrix} = \begin{pmatrix}1\\0\\1\end{pmatrix}$$

$$\underline{3} = \underline{1} \oplus \underline{2}\,.$$

$$|u\rangle = \frac{|1\rangle + |2\rangle + |3\rangle}{\sqrt{3}} \xrightarrow{D} |u\rangle . \quad \in \underline{1}\,.$$



$$D_n = \langle a, b \mid a^n = e,\ b^2 = e,\ bab = a^{-1} \rangle$$

$$\supset \mathbb{Z}_n = \langle a \mid a^n = e \rangle .$$

$$H = D(a) + D(a)^\dagger \quad \text{on} \quad \mathbb{1}$$

evals of $D(a)$: decompose into irreps

$$D(a) = \sum_k \omega^k |k\rangle\langle k|$$

$$|k\rangle = \frac{1}{n} \sum_{j=1}^{n} \omega^{-kj} |j\rangle \qquad k = 1..n .$$

$\Rightarrow$ evals are $\quad 2\cos 2\pi k/n,\ k = 1..n.$

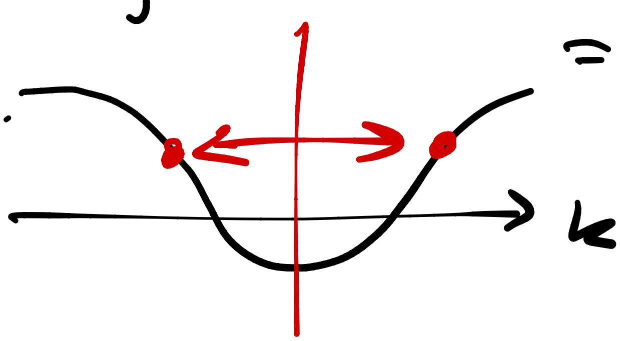
$$D(b)|j\rangle = |-j\rangle = |n-j\rangle$$

$$D(b)|k\rangle = \frac{1}{n} \sum_{j=1}^{n} \omega^{-kj} |n-j\rangle = \frac{1}{n} \sum \omega^{+kj} |j\rangle$$

$$= |-k\rangle .$$

$[H, D(b)] = 0$.

$\Rightarrow$



$\epsilon(k) = \epsilon(-k).$

given R, $D_R(g)$.

$\bar{R}$: $\underline{D_{\bar{R}}(g) \equiv D_R(g)^*}$.

$$D(g) = \sum u_\lambda(g) \underline{|\lambda\rangle\langle\lambda|}$$

each $D(g)$ has $D(g)D(h) = D(h)D(g)$ $\forall g \in G$.

$\xrightarrow{\text{Schur}}$ on an irrep

$D(g) = u(g)\mathbb{1}$.

$\underline{M_{ab}{}^c}$