Group presentation:

$$G = \langle a_1 \dots a_k \mid \underline{r_1, r_2 \dots} \rangle$$
$$= \{e, a_1, a_2, \dots, a_k, a_1^{-1}, a_2^{-1}, \dots, a_1a_2, a_2a_1, a_1a_2^{-1} \dots\} /\sim$$

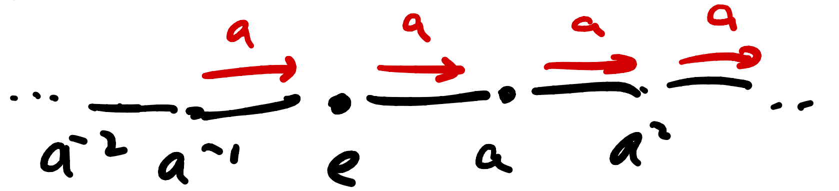
Visualization (Cayley diagram):

vertices $\leftrightarrow$ elements of G

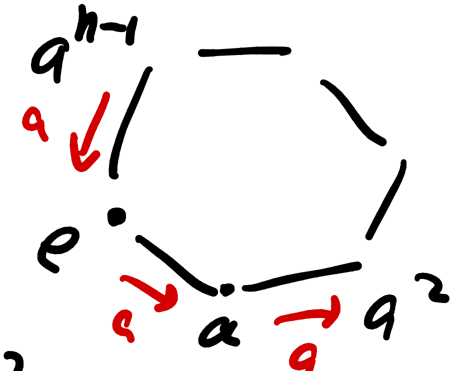
edges $\leftrightarrow$ generators

loops $\leftrightarrow$ relations.

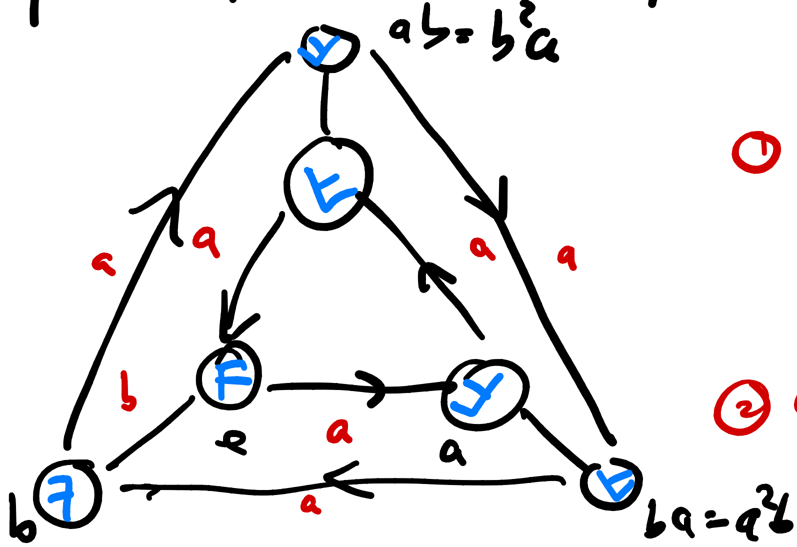
$$\mathbb{Z} = \langle a \mid \ \rangle$$



$$\mathbb{Z}_n = \langle a \mid a^n = e \rangle$$



$$D_n = \langle a, b \mid a^n = e, b^2 = e, (ab)^2 = e \rangle$$

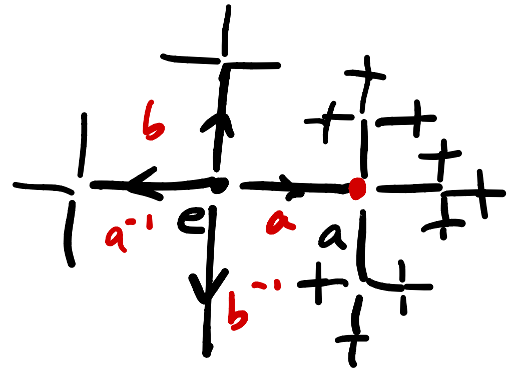


Warnings:

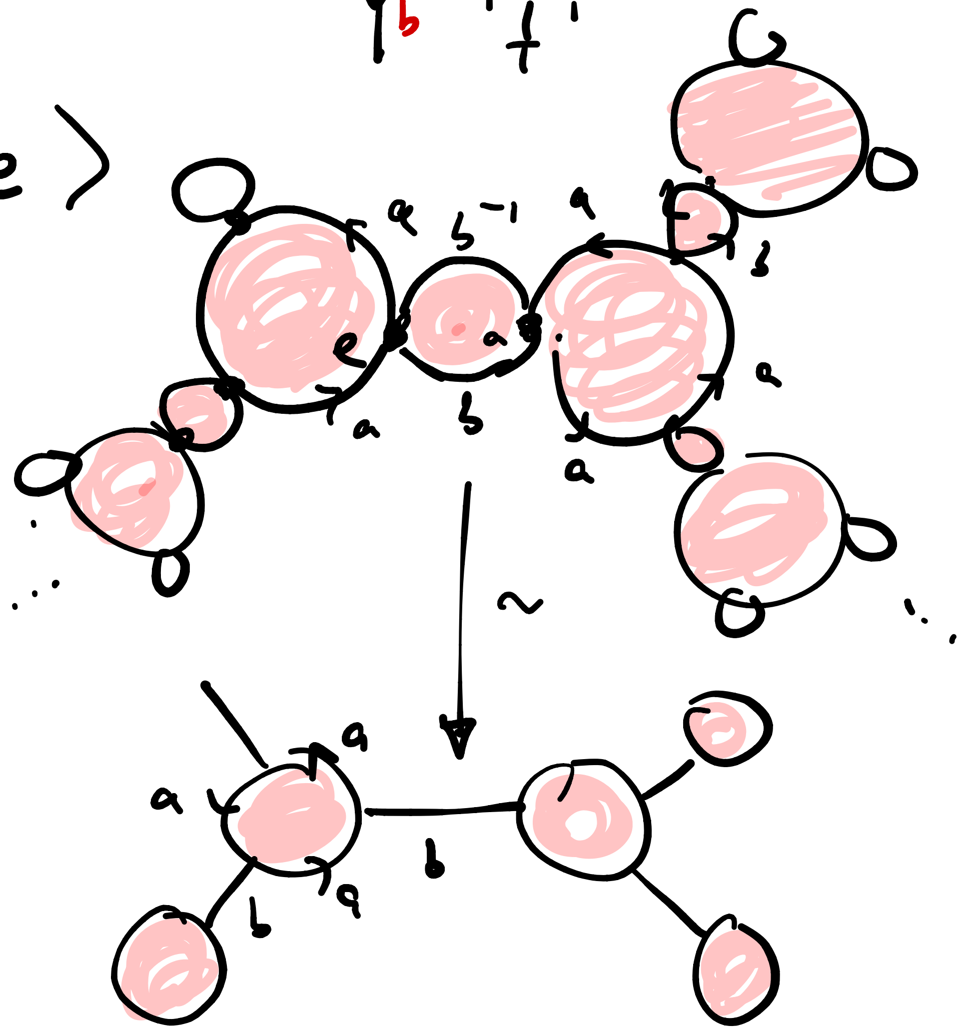
① Some people call "D_n", "D_{2n}"

② Cayley diagram depends on presentation.

$\langle a, b \mid \quad \rangle$:



$\langle a, b \mid a^3 = e, b^2 = e \rangle$



Find:

X_G s.t. $\pi_1(X_G) = G$.

$Y \equiv$ Attach to each loop of Cayley diag of G a disk $\Big\}$ has $\pi_1(Y) = \{e\}$

$X_G = Y/G$.

1.3 Subgroups & Conjugacy classes

Cosets. Given a subgroup $H < G$

$H = \{ h_1, \dots h_n \}$ $\quad n = |H|$

$h_1 = e$.

coset of H by g_1: $\quad g_1 H \equiv \{ g_1 h_1, \dots g_1 h_n \}$

$|g_1 H| = n$

For $g_2 \neq g_1 \in G$ either $g_1 H \cap g_2 H = \phi$

or $g_1 H = g_2 H$.

pf: If $g_1 h_1 = g_2 h_2 \Rightarrow g_1 = g_2 \underbrace{h_2 h_1^{-1}}_{\in H} \in g_2 H$

$\Rightarrow \quad G = g_1 H \,\dot\cup\, g_2 H \,\dot\cup\, g_3 H \dots \dot\cup\, g_k H$

for some $k \in \mathbb{Z}$.

Thm (Lagrange): $|H| k = |G|$

$k = |G| / |H|$ "index of subgp". $\{\text{cosets}\} = G/H$.

An invariant or normal subgroup H:

$$g^{-1} H g = H \quad \forall g \in G.$$

($\text{ie } g^{-1} h g \in H \quad \forall g \in G,\ h \in H.$)

If H is normal, G/H is a group. ["quotient group".]

$(g_i H)(g_j H) \equiv$ the coset $g_k H$ containing $g_i h_i g_j h_j$

$= g_i g_j$

Pick a representative of each coset

$g_i h_i \quad , g_j h_j$

$$g_i h_i g_j h_j = g_i g_j \underbrace{g_j^{-1} h_i g_j}_{= h_k \text{ if } H \text{ is normal}} h_j \in g_i g_j H$$

$e = g_j g_j^{-1}$

Q: How tell if $H < G$ is normal?

G is simple if it has no inv't subgroups.

Conjugacy classes: $g_1, g_2 \in G$ are conjugate if $g_1 = g^{-1} g_2 g$ for some $g \in G$.

Conjugation is an equivalence rel'n

(reflexive $g \sim g$, symmetric $g_1 \sim g_2 \Leftrightarrow g_2 \sim g_1$,

transitive $g_1 \sim g_2 , g_2 \sim g_3 \Rightarrow g_1 \sim g_3$)

$[g_1] = [g_2]$ if $g_1 \sim g_2$

$C_g \equiv \{ k g k^{-1} , k \in G \}$. conjugacy class of g.

$$G = C_e \dot{\cup} C_{g_1} \dot{\cup} C_{g_2} \cdots$$

- $C_e = \{e\}$
- If G is abelian $C_g = \{ k g k^{-1} \} = \{g\}$

claim: $|C_g| \equiv$ # of el'ts of C_g divides $|G|$.

("orbit-stabilizer thm")

$Z_g \equiv \{ h \in G \mid h^{-1} g h = g \}$ "centralizer of g"

i.e. $gh = hg$

is a subgroup of G.

Iff $k \in Z_g$, $k g k^{-1} = g$ is not a new element of C_g.

$$\Rightarrow |C_g| = |G| / |Z_g| .$$

examples: • $R(\hat{n},\theta) \sim R(\hat{n}',\theta)$ in $SO(3)$

$$R(\hat{n},\theta) = g\, R(\hat{n}',\theta)\, g^{-1}$$

w/ g = rotation taking $\hat{n}'$ to $\hat{n}$.

• conjugacy classes of S_n: labelled by cycle structure

pf outline: – any perm is a product of exchanges (ij)

– conjugation by (ij) interchanges the positions of i & j

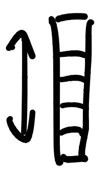
$$(ij)\,(i\,3\,5)(j\,2\,7\,6)\,(ij)^{-1}$$

$$= (j\,3\,5)(i\,2\,7\,6)$$

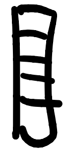
A cycle structure of $\pi \in S_n$: k_j cycles of length j

$$\sum_j j k_j = n$$

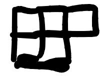
"partition of n"

Young diagram (Ferrers diagram): j  for each j-cycle, order by decreasing ht. top-justify

S_5 :

$(12345) \longleftrightarrow$ 

1 5-cycle

$(12)(34)(5) \longleftrightarrow$ 

$k_1 = 1,\ k_2 = 2$

$(1)(2)(3)(4)(5) \longleftrightarrow$ 

$k_1 = 5$

$n = 5$ boxes.

$$\#\text{ of elements of } \{k_j\} = \frac{n!}{\prod_j j^{k_j} k_j!} = \frac{|S_n|}{|Z_{\{k\}}|}$$

cyclic perms within cycles

perms of cycles of same length.

$Z\big((12)(34)(5)\big) = ?$

$(1 \leftrightarrow 2),\ (3 \leftrightarrow 4)$

$12 \longleftrightarrow 34$

1.4 Group Actions

A group action (of G) on a finite set X is a

map $\quad G \times X \longrightarrow X$

$$(g, x) \longmapsto gx \equiv g(x) \equiv \bar{g}x$$

that respects the product on G : $(hg)x = h(gx)$ ⟹ must be invertible!

group action : rep. thy

classical mech : Q.M.

allow superpositions

eg: G acts on G (in many ways)

$$G \times G \to G$$
$$(g_1, g_2) \longmapsto g_1 g_2$$

is a group action in which g_1 is a perm on $|G|$ elements.

Any $G \times X \to G$ realizes G as a subgroup of S_n, $n = |X|$.

Orbits: each $x \in X$ is part of an **orbit**

$$Gx \equiv \{ gx,\ g \in G \} \quad \text{orbit of } x$$

$$G_x \equiv \{ g \in G \text{ s.t. } gx = x \} \quad \text{stabilizer subgroup } \subset G \text{ of } x$$

claim: (orbit-stabilizer thm)

$$|G| = |G_x|\,|Gx| \quad \forall x \in X.$$

Pf: let $Gx = \{ \underline{x_1, x_2 \ldots x_k} \}$ $\quad x_1 = x$

$k = |Gx|$.

$\forall g$, $gx = x_i$ for some i

let $G_i \equiv \{ g \in G \mid gx = x_i \}$ $\quad i = 1..k$

$$G = G_1 \dot\cup G_2 \dot\cup \cdots \dot\cup G_k \qquad (G_1 = G_x.)$$

claim: $|G_x| = |G_i|$ $\quad i = 1..k$.

pf of claim: bijection $\phi : G_x \to G_i$

$g \mapsto \underbrace{h_i g}_{\in G_i \text{ because}} = \phi(g)$

$h_i g x = h_i x = x_i$

pick $h_i \in G_i$

i.e. $\underline{h_i(x) = x_i}$

$$\boxed{\begin{aligned} \phi^{-1} &: G_i \to G_x \\ g &\mapsto h_i^{-1} g \end{aligned}}$$

$\Rightarrow |G| = k|G_x| = |Gx||G_x|$.

<u>example</u>: $X = G$ with action by conjugation.

$$G \times G \longrightarrow G$$
$$(g, x) \longmapsto gxg^{-1}$$

orbits $Gx = C_x$ conjugacy classes $x \in G$.

stabilizer subgroup $G_x = Z_x$ centralizer of x

$|G|/|G_x| = |Gx| = $ # of elts of conjugacy class.

$G_i \longleftrightarrow$ cosets of Z_x by x_i

elements of conjugacy class.

<u>Note</u>: $Gx = Ggx$ $\forall g \in G$ $\Rightarrow$ two orbits are identical or disjoint.

$\{\text{distinct orbits}\} \equiv X/G$ "X modulo G".

$$X = \bigcup_{O \in X/G} \{x \in O\}.$$