**Bestiary:** • Integers $\mathbb{Z}$ under $+$ : discrete

$(k,l) \longmapsto k+l$. : infinite

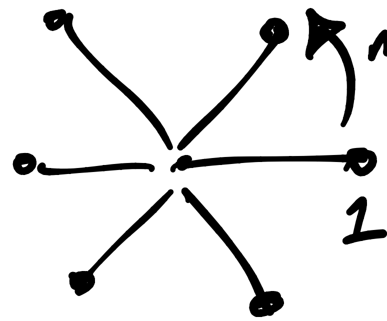
: non-compact

• Integers modulo $n$ under $+$ $\equiv \mathbb{Z}_n$ · discrete

$(k,l) \longmapsto (k+l)_n \equiv k+l$ modulo $n$ · finite

let $\omega \equiv e^{2\pi i/n}$

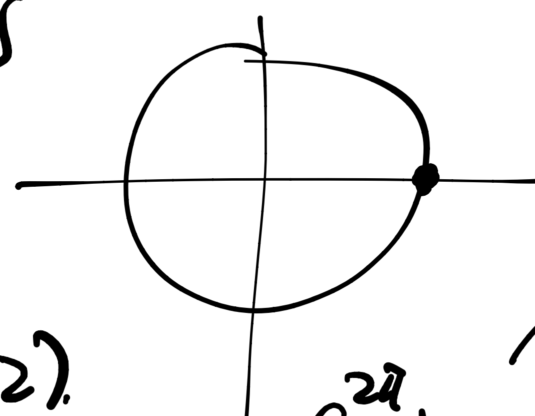
$\omega^k \omega^l = \omega^{k+l}$




mult by $\omega$ = rot. by $2\pi/n$.

• Take $n \to \infty$

$U(1) = \{ e^{i\phi},\ \phi \in [0, 2\pi) \}$



infinite
Continuous
compact

$e^{i\phi} = \cos\phi + i\sin\phi$

$U(1) =$ rotations in 2d $= SO(2)$.

$$\int_0^{2\pi} d\phi\, 1 = 2\pi.$$

• real #s $\mathbb{R}$ under $+$ $\infty$ locally $\mathbb{R}^n$

continuous

non-compact.

An (infinite) continuous group which is also a manifold is a Lie group. eg: $U(1) = SO(2)$, $\mathbb{R}$.

Non-abelian example: • rotations in $\mathbb{R}^3 = SO(3)$.

• rotations by multiples of $\pi/2$ in $\mathbb{R}^3$. (finite) (∞ continuous compact)

$\equiv O$ OR $A_4$.

Matrix Lie groups: invertible matrices $n\times n$ under matrix multiplication

$\equiv GL(n, F)$

matrix elements $\in F = \mathbb{Z}, \mathbb{R}, \mathbb{C}$

$\{ n\times n\ M \text{ w/ } \underline{\underline{\det M = 1}} \} \equiv SL(n, F)$

$\det(M_1 M_2) = \det M_1 \det M_2$.

noncompact

$$O(n) = \{ n\times n \text{ real matrices s.t. } O^T O = \mathbb{1} \}$$

$$O^i{}_k\, \delta_{ij}\, O^j{}_\ell = \delta_{k\ell}$$

$$\delta_{k\ell} \equiv \begin{cases} 1 & \text{if } k=\ell \\ 0 & \text{else} \end{cases}$$

vector:

$v^i \mapsto O^i{}_j v^j$ then $\|v\|^2 = v^T v \mapsto v^T O^T O v$

$= v^T v = \|v\|^2$.

$$\begin{pmatrix}x\\y\\z\end{pmatrix} \to \begin{pmatrix}-x\\y\\z\end{pmatrix} = \underbrace{\begin{pmatrix}-1&&\\&1&\\&&1\end{pmatrix}}_{\det = -1}\begin{pmatrix}x\\y\\z\end{pmatrix}$$

$$SO(n) = \{ M \in O(n) \text{ s.t. } \det M = 1 \}.$$

$$1 = \det 1 = \det O^T O = (\det O)^2 \to \det O = \pm 1.$$

$O(n) =$


$\det = -1$ $\qquad$ $\det = +1$

SO(n)

eg: $O \in SO(3)$ : $\begin{pmatrix}\cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}$

$$O(n) \stackrel{?}{=} SO(n) \times \mathbb{Z}_2$$

---

Replace $\delta_{ij}$ m $\eta_{\mu\nu} = \left(\begin{array}{c|c} \mathbb{1}_{n\times n} & 0 \\ \hline 0 & -\mathbb{1}_{m\times m}\end{array}\right)_{\mu\nu}$

$$\{ L^\mu{}_\nu \mid L^T \eta L = \eta \} = O(n, m)$$

noncompact for $n, m \geq 1$

$$Sp(n) = \{ \underset{\substack{n\times n\\ \text{real}}}{M} \mid M^T \mathcal{E} M = \mathcal{E} \} \qquad \mathcal{E} = \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & 0 & 1 & \\ & & -1 & 0 & \\ & & & & \ddots \end{pmatrix}$$

$$\{f, g\}_{PB} = \partial_q f \, \partial_p g - \partial_p f \, \partial_q g .$$

(compact)

---

$U : |\psi\rangle \mapsto U|\psi\rangle$ preserving

$$\langle\phi|\psi\rangle \overset{!}{=} \langle\phi|U^\dagger U|\psi\rangle \quad \forall \phi, \psi$$

$$\Rightarrow \quad U^\dagger U = \mathbb{1}$$

$$U(n) = \{ U \ n\times n, \text{ complex s.t. } U^\dagger U = \mathbb{1} \}.$$

(compact)

<u>Fact</u>:

compact Lie groups = (products of)

$SO(n), Sp(n), U(n), E_{6,7,8}, F_4, G_2.$

(!)

# FINITE GROUPS.

# Symmetric group $S_n$ $\quad |S_n| = n!$

$\{1, \dots n\} \longrightarrow \{\pi_1, \dots \pi_n\}$

$\pi = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ \pi_1 & \pi_2 & \pi_3 & & \pi_n \end{pmatrix}$. $\quad e = \begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & 2 & \cdots & n \end{pmatrix}$

(object in location $i$ $\rightarrow$ location $\pi_i$) $\quad \pi^{-1} = \begin{pmatrix} \pi_1 & \pi_2 & \cdots & \pi_n \\ 1 & 2 & \cdots & n \end{pmatrix}$

eg: $\underbrace{\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}}_{(13)} \underbrace{\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}}_{(23)} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \\ 3 & 1 & 2 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}}_{(132)}$

follow one element through $\underline{\pi^k}$. = a cycle

$e = (1)(2) \cdots (n)$

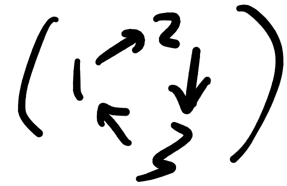
$\begin{pmatrix} 1 & 2 & 3 & \cdots \\ 2 & 1 & 3 & \cdots \end{pmatrix} = (12)(3) \equiv \underline{(12)}$ interchange

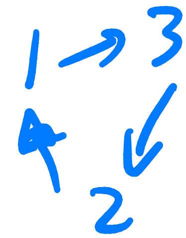
$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 2 & 1 \end{pmatrix} = (135)(24)$

**Fact**: every perm = product of 2-cycles (interchange)

# Rules for multiplying cycles:

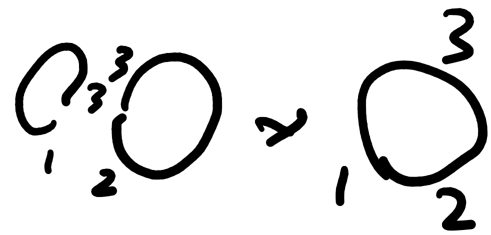
- $(12) = (21)$ $\quad$ $(123) = (231) \neq (132)$

  "$\equiv$  "



- $(12)^2 = e$.

- $(13)(32) = (132)$



- $(12)\underbrace{(234)}_{=(23)(34)} = (1234)$

---

given $G_1$ & $G_2$ $\longrightarrow$ $G_1 \times G_2 =$

$$\left\{ (g_1, g_2) \;\; \begin{matrix} g_1 \in G_1 \\ g_2 \in G_2 \end{matrix} \right\}$$

$$(g_1, g_2) \cdot (g_1', g_2) = (g_1 g_1', g_2 g_2')$$

A subset $H$ of $G$ which is itself a group is a subgroup $H \subset G$.

eg: $g \in G$. $G \supset \{e, g, g^2 \dots g^{n-1}\} \cong \mathbb{Z}_n$.

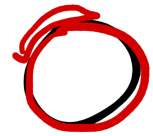
## Point groups : "$C_{3v}$, $T$, $O$, $I$ ..."

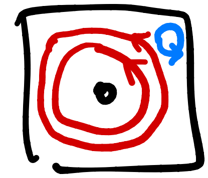
are products of $\mathbb{Z}_n$, $S_n$ ...

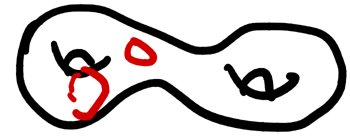
## Fundamental groups of spaces

Given $X$

define a group.

eg: $S^1 =$ 

$\mathbb{R}^2 \setminus \{0\}$ 

$\Sigma_2$ 

$f : S^1 \to X$ continuous

$f(0) = f(2\pi) = p.$

$$(f \cdot g)(\theta) = \begin{cases} f(2\theta) & \theta < \pi \\ g(2(\theta - \pi)) & \theta > \pi \end{cases}$$

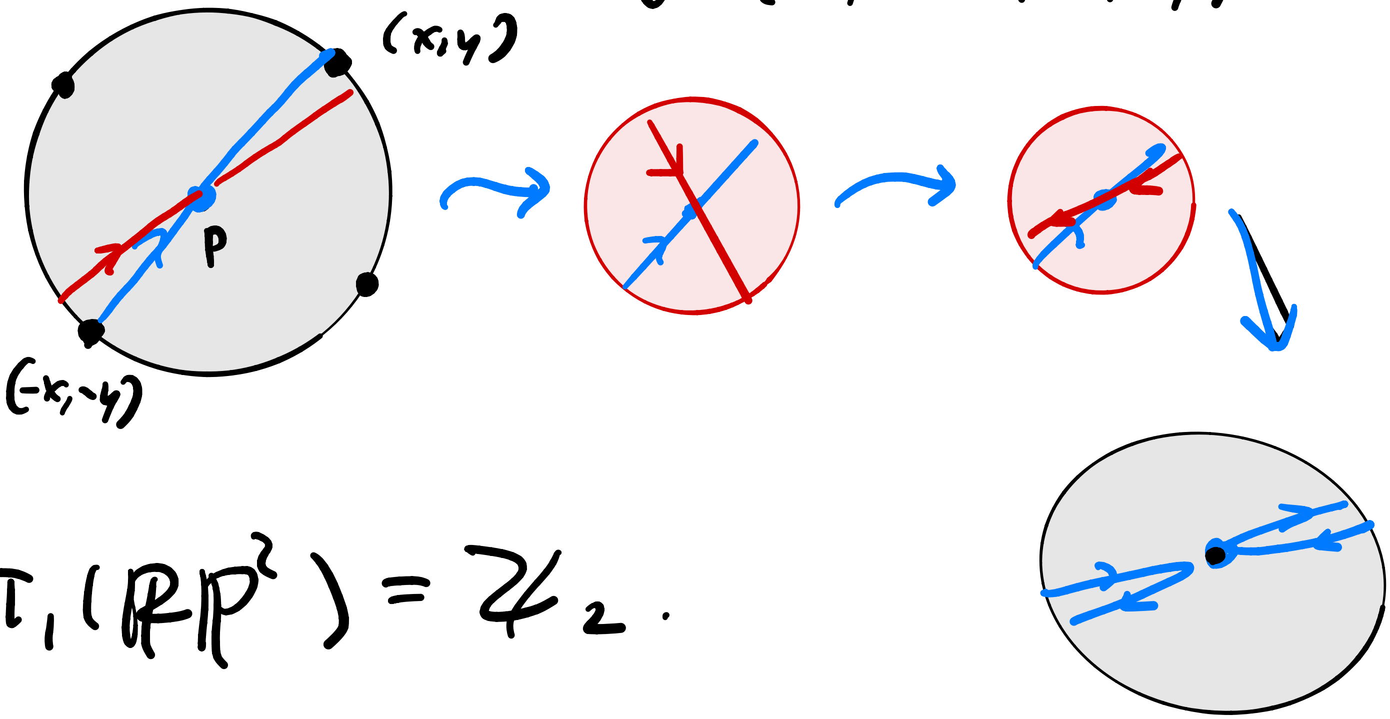
$e = \{ f(\theta) = p \}$. $\quad f^{-1}(\theta) = f(2\pi - \theta)$.

$$\pi_1(X) = \{ f : S^1 \to X \ldots \} / \sim$$

$f \sim g$ if $\exists$ a family of such maps which interpolates from $f$ to $g$.

$\pi_1(S^1) = \pi_1(\mathbb{R}^2 \setminus \{0\}) = \mathbb{Z}$ winding #.

$\mathbb{R}\mathbb{P}^2 \equiv$ disk / pts on the boundary identified by $(x,y) \to (-x,-y)$.




$\pi_1(\mathbb{R}\mathbb{P}^2) = \mathbb{Z}_2$.

---

**Embedding in $S_n$** $\quad G = \{g_1, g_2, \dots g_n\}$

$h \in G$: $\quad G = \{hg_1, hg_2 \dots hg_n\}$

$\quad = \{g_{\pi_1}, g_{\pi_2} \dots g_{\pi_n}\} \quad \pi \in S_n$

Any finite $G \subset S_n$ for $n \leq |G|$.

[Cayley].

**Fact:** The cycle lengths of $\pi$ are all THE SAME.

Pf: Suppose otherwise eg $\pi = (123)(45) = (45)(123)$

$\pi^2 = (132)$ Fixes 4 & 5 forbidden by Sudoku.

$\Rightarrow$ if $|G| = p$, prime $\Rightarrow G = \mathbb{Z}_p$.

eg: $(123)(34) = (1234)$.

cycle decomposition $\equiv$ decomp into non-overlapping cycles

$(123)(456)$ ✓

$\left((123)(4567)\right)^3 = (7154)$ ✗.

---

<u>Generators & Relations</u>:

$$\mathbb{Z}_n = \langle \underset{\text{generators}}{r} \mid \underset{\text{relations}}{r^n = e} \rangle = \{\text{all powers \& products of generators \& } e\} / \text{rels}$$

$$S_3 = \langle r, s \mid r^3 = e, s^2 = e \rangle$$

$r = (123) \quad s = (12)$

(dihedral)

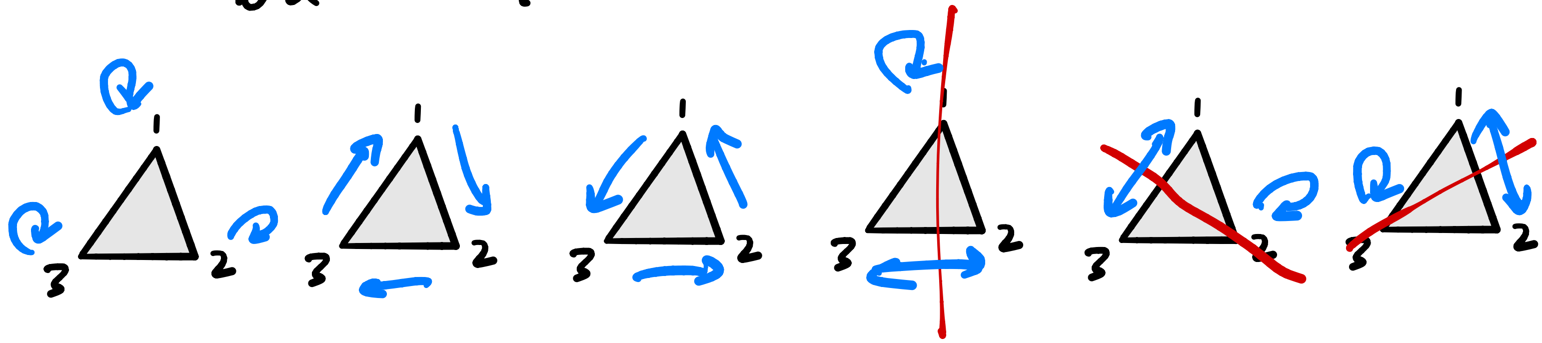
$$D_n = \langle a, b \mid a^n = e, b^2 = e, (ab)^2 = e \rangle$$

symmetry gp of regular n-gon.

$a$ = rot. by $2\pi/n$    $b$ = reflection in a plane

$\underbrace{\phantom{xxxx}}_{a^n = e}$      $b^2 = e$

$bab^{-1} = a^{-1}$   mirror reverses dir. of rotation.

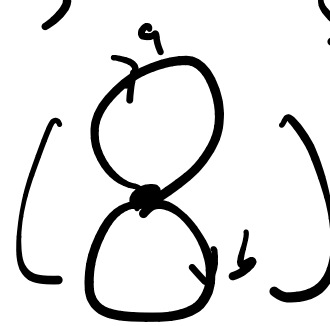


| | | | | | |
|---|---|---|---|---|---|
| $e$ | $a$ | $a^2$ | $b$ | $ba$ | $ba^2$ |
| $e$ | $(123)$ | $(132)$ | $(32)$ | $(13)$ | $(12)$ |

$D_3 = S_3$.

cf: $\langle a \mid \cdot \rangle = \mathbb{Z}$.

$\langle a, b \mid \cdot \rangle = \{e, a, b, ab, ba, a^{-1}, b^{-1}, a^2b, aba, \dots\}$

$= F_2$ free group on 2 elements. $= \pi_1\big($  $\big)$

presentations are not unique:

$$D_4 = \langle b, c \mid b^2 = c^2 = e,\ bcbc = cbcb \rangle$$

$bc =$ rotation.

$$\Gamma_{\text{english}} \equiv \langle a, b, \dots z \mid A = B \text{ if the words } A \& B \text{ are homophones} \rangle$$

$$= \{e\} = \Gamma_{\text{French}} \neq \Gamma_{\text{Japanese}}.$$

$$\det\begin{pmatrix} -1 & & \\ & -1 & \\ & & -1 \end{pmatrix} = -1 \qquad \det\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = +1$$

$$\det\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = -1$$

$G_1 \ltimes G_2$ "semi-direct product"

$$= \{ (g_1, g_2),\ g_1 \in G_1,\ g_2 \in G_2 \}$$

$$\underbrace{(g_1, g_2)} \cdot \underbrace{(g_1', g_2')} \qquad (g_1', g_2') \cdot \underbrace{(g_1, g_2)}_{\in G_1 \ltimes G_2}$$

e.g. $= (g_1 g_1',\ D(g_1)\, g_2 g_2')$

some action of $g_1$ on $g_2$

eg:

$$E(2) = \{\text{transl \& rot of } \mathbb{R}^2\}$$

$$\overset{?}{=} SO(2) \ltimes \mathbb{R}^2$$

$$\overset{?}{=} \mathbb{R}^2 \rtimes SO(2)$$