University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 220 Symmetries Fall 2020
Assignment 9

Due 12:30pm Monday, December 7, 2020

Thanks for following the submission guidelines on hw 01. Please ask me by email if you have any trouble.

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1. **Brain-warmer.** Consider the adjoint of $\mathsf{SU}(3)$ with highest weight state $|\mu^1 + \mu^2\rangle = |(1,1)\rangle$. Check that the two states with weight zero $|A\rangle \equiv E_{-\alpha^1} E_{-\alpha^2} |(1,1)\rangle$ and $|B\rangle \equiv E_{-\alpha^2} E_{-\alpha^1} |(1,1)\rangle$ are linearly independent (in agreement with the fact that there are two Cartan generators).

   Hint: show that two states $|A\rangle$ and $|B\rangle$ are linearly dependent only if $\langle A|A\rangle \langle B|B\rangle = \langle A|B\rangle \langle B|A\rangle$.

2. **Representations of G_2.**

   In this problem we'll build representations of G_2 from scratch (*i.e.* from the Dynkin diagram). With enough effort we can see that G_2 is a subgroup of $\mathsf{SO}(7)$ and that it has an antisymmetric cubic invariant.

   (a) Check that the simple roots

   $$\alpha^1 = (0,1), \ \alpha^2 = (\sqrt{3}, -3)/2$$

   reproduce the Dynkin diagram of G_2.

   (b) Find the fundamental weights μ^a of G_2. Show that they are also roots! In particular you should find that $\mu^1 = 2\alpha^1 + \alpha^2, \mu^2 = 3\alpha^1 + 2\alpha^2$. This means that the root lattice and the weight lattice are the same in this case.

   (c) Find the orbit of μ^1 under Weyl reflections, and thereby draw the weight diagram for the representation with highest weight μ^1, $R_{(1,0)}$ (I recommend some symbolic software). Starting from the highest weight state, find a path connecting all these weights, where each step moves by (minus) a simple root. Conclude that $(0,0)$ must also be a weight vector. You should find that $R_{(1,0)} = \mathbf{7}$ is 7 dimensional. This is the fundamental representation of G_2 in the sense that all other reps appear in its tensor products.

   Bonus: label the weights by their $p^a - q^a$ vectors and check that the decompositions into $\mathsf{SU}(2)_{\alpha^a}$ multiplets makes sense.

(d) Find the orbit of μ^2 under Weyl reflections and draw the weight diagram for the representation with highest weight μ^2, $R_{(0,1)}$. Conclude that $R_{(0,1)} = \mathbf{14}$ is the adjoint rep of G_2.

(e) When we take tensor products, what happens to the weights? That is, given two reps $\mathbf{a}$ and $\mathbf{b}$ with highest weight vectors $\mu_{\mathbf{a}}$ and $\mu_{\mathbf{b}}$ respectively, what is the highest weight vector of $\mathbf{a} \otimes \mathbf{b}$?

(f) What is the highest weight vector of $\Lambda^2\mathbf{7}$? (Hint: the highest weight vector of $V \otimes V$ is symmetric under interchange of the two factors).

(g) [Bonus problem] Draw the weight diagram for $\Lambda^2\mathbf{7}$. Conclude that $\Lambda^2\mathbf{7} = \mathbf{7} \oplus \mathbf{14}$.

(h) [Bonus problem] Draw the weight diagram for $\mathrm{Sym}^2\mathbf{7}$. Conclude that $\mathrm{Sym}^2\mathbf{7}$ contains a copy of $R_{(2,0)}$, whose dimension we don't know yet. By counting the multiplicity of the $(0,0)$ weight vector, show that $\mathrm{Sym}^2\mathbf{7} = R_{(2,0)} \oplus \mathbf{1}$. Conclude that $G_2 \subset \mathsf{SO}(7)$.

(i) [Bonus problem] Draw the weight diagram for $\Lambda^3\mathbf{7}$. Show that $\Lambda^3\mathbf{7} = R_{(2,0)} \oplus \mathbf{7} \oplus \mathbf{1}$. Conclude that G_2 has an antisymmetric cubic invariant. In fact G_2 can be defined as the subgroup of $\mathsf{SO}(7)$ which preserves an antisymmetric 3-index tensor.

[Cultural remark: this also means that it preserves a spinor of $\mathsf{SO}(7)$. For this reason, 7-manifolds with G_2 holonomy admit a covariantly-constant spinor (the generic orientable 7-manifold has holonomy $\mathsf{SO}(7)$). Compactification of supersymmetric field theories (such as 11-dimensional supergravity) on such manifolds therefore preserves some supersymmetry.]

(j) [Bonus problem] Show that the irrep with highest weight $a\mu_1 + b\mu^2$ with arbitrary $a, b \in \mathbb{Z}_{\geq 0}$ (*i.e.* the most general possible representation) is contained in the tensor product $\mathbf{7}^{\otimes n}$ for some n.

3. **Geometry problem.** [Bonus problem] Show that the sum of the three angles between three linearly independent vectors in $\mathbb{R}^3$ is less than 2π.

4. **$\mathsf{SO}(5)$ and $\mathsf{Sp}(4)$.**

(a) The simple roots of $\mathsf{so}(2n+1)$ are $e^i - e^{i+1}, i = 1..n-1, e^n$. Find the fundamental weights of $\mathsf{so}(5)$, μ^1 and μ^2. Build the weight diagrams for the two representations R_{μ^1} and R_{μ^2}.

(b) The simple roots of $\mathsf{sp}(2n)$ are $e^i - e^{i+1}, i = 1..n-1, 2e^n$. Find the fundamental weights of $\mathsf{sp}(4)$, μ^1 and μ^2 Build the weight diagrams for the two representations R_{μ^1} and R_{μ^2}.

(c) Compare.

(d) Argue that the anti-symmetric square of the spinor rep of $\mathsf{so}(4)$, $\Lambda^2 \mathbf{4}$ contains a singlet.

5. **Spinor reps.**

(a) Find the constant $C(n)$ such that

$$\gamma_F \equiv C(n)\gamma_1 \cdots \gamma_{2n}$$

satisfies

$$\gamma_F = \gamma_F^\dagger \quad \text{and} \quad \gamma_F^2 = 1.$$

(Here γ_i are hermitian Majorana operators, satisfying $\{\gamma_i, \gamma_j\} = 2\delta_{ij}$.)

(b) Check that $T_{a,2n+1} = -ST^\star_{a,2n+1}S^{-1}$ Conclude that the spinor rep of $\mathsf{SO}(2n+1)$ is not complex (where S is given in the lecture notes).

The following problems I'll postpone until the next problem set. I leave them here too in case you started working on them and can't stop.

6. **Ramond-Ramond sectors.** [Bonus problem]

(a) The *Ramond* sector of the superstring worldsheet contains a Hilbert space on which 8 majorana operators γ_i act. The $\mathsf{SO}(8)$ acting on the index i in the fundamental is part of the spacetime symmetry. Physical states of the superstring are those which have definite eigenvalue of $\gamma_F = C\prod_{i=1}^{8}\gamma_i$ (where C is chosen so that $\gamma_F^\dagger = \gamma_F$ and $\gamma_F^2 = 1$).

The Ramond-Ramond sector of the closed superstring Hilbert space is the tensor product two Ramond sectors (one from right-moving modes and one from the left-moving modes on the closed string worldsheet). In type IIB, both copies have the same eigenvalue of γ_F and in type IIA, the two copies have opposite γ_F eigenvalue.

How do the physical states of each type of closed superstring transform under $\mathsf{SO}(8)$?

Removing the string theory jargon, the question is: how do $\mathbf{8}_+ \otimes \mathbf{8}_+$ and $\mathbf{8}_+ \otimes \mathbf{8}_-$ decompose into irreps of $\mathsf{SO}(8)$?

One way to do it is to consider the transformation law for objects of the form $\langle s_1 s_2 s_3 s_4 | \gamma^{i_1}\gamma^{i_2} \cdots \gamma^{i_k} | s_1 s_2 s_3 s_4 \rangle$.

(b) [Super-bonus problem – requires some field theory] Consider the action

$$S[X^i, \psi^i] = \int d^2x((\partial X)^2 + \psi \not{\partial} \psi),$$

where $i = 1..n$. Take space to be periodic $x \equiv x + L$ and take periodic (Ramond) boundary conditions on the fermions fields (and the scalars). Show that its groundstates form a tensor product of two spinor representations of $\mathsf{SO}(n)$.

7. **Schwinger bosons.**

What happens if in our construction of spinor reps, we replace the fermions $\{c_a, c_b^\dagger\} = \delta_{ab}$ with bosons $\{b_a, b_b^\dagger\} = \delta_{ab}$?

(a) First consider what representations this produces of the $\mathsf{SU}(n)$ subalgebra

$$H_a = b_a^\dagger b_a - \frac{1}{2}, \quad E_{ab} = b_a^\dagger b_b, a \neq b.$$

Hint: consider states of fixed particle number $\sum_a H_a = k$.

I recommend starting with the case of $n = 2$.

(b) Can you make representations in this way of the full $\mathsf{SO}(n)$ algebra which includes

$$E'_{ab} = b_a^\dagger b_b^\dagger, a \neq b.$$

What about $b_a^\dagger b_b^\dagger$ with $a = b$?