University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 220 Symmetries Fall 2020
Assignment 6

Due 12:30pm Monday, November 16, 2020

Thanks for following the submission guidelines on hw 01. Please ask me by email if you have any trouble.

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1. **Non-degeneracy of braiding in the 3d quantum double model.**

   The title of this problem is a description of the physics situation where I (just recently) encountered the following nice group theory fact. You don't need to know what it means to do the problem!

   (a) Prove the following identity using character orthogonality: for any finite group G,

   $$\sum_{\alpha} |Z_\alpha|^{n-2} = \sum_{a_1 \cdots a_n} N^{1}_{a_1 \cdots a_n} N^{1}_{\bar{a}_1 \cdots \bar{a}_n}$$

   where Z_α is the centralizer of an element of conjugacy class α, $a_1 \cdots a_n$ are irreps of G, and $N^1_{a_1 \cdots a_n}$ is the number of times $R_{a_1} \otimes R_{a_2} \otimes \cdots R_{a_n}$ fuses to the identity[1]. The symbol $\sum_{a_1 \cdots a_n}$ means n independent sums over all the irreps of G.

   [Cultural remark about the title of the problem, ignore if you want: The quantum double model is an exactly solvable lattice model with topological order. There is one for every finite group, and it can be put on any lattice. If we put this model on a 3d lattice, the LHS of this formula has an interpretation as the number of excitations shaped like an n-hole donut. The RHS has an interpretation as the number of collections of particles which can be braided non-trivially around such an object. The idea of "braiding non-degeneracy" in a system with topological order is that each topologically non-trivial excitation should be detectable from a distance by moving some *other* excitation around it.]

   (b) What is N^1_{abc} for the group S_3? Find a group where N^1_{abc} takes a value other than 0 or 1 for some choice of abc.

[1] More precisely, $N^1_{a_1 \cdots a_n}$ is the dimension of the 'fusion space' $V^1_{a_1 \cdots a_n}$ in

$$R_{a_1} \otimes R_{a_2} \otimes \cdots R_{a_n} = \mathbf{1} \otimes V^1_{a_1 \cdots a_n} \oplus \cdots$$

where $\mathbf{1}$ is the trivial rep of G.

2. **Control-Z algebra.** [Bonus problem]

   Recall that

$$\mathsf{CZ}_{ij} \equiv e^{\mathbf{i}\frac{\pi}{4}(1-Z_i)(1-Z_j)} = \mathsf{CZ}_{ji}$$

   is the control-Z operation.

   (a) Check that in the Z basis $(Z\,|s\rangle = s\,|s\rangle\,, s = 0,1)$ CZ acts as Z on the second bit only if the first bit is 1:

$$\mathsf{CZ}\,|0s\rangle = |0s\rangle\,, \mathsf{CZ}\,|1s\rangle = (-1)^s\,|1s\rangle\,, \quad \bar{0} \equiv 1, \bar{1} \equiv 0.$$

   (b) Check that

$$X_i \mathsf{CZ}_{ij} X_i = \mathsf{CZ}_{ij} Z_j. \tag{1}$$

3. **Practice at exponentiating matrices.**

   (a) Check that the matrices $(J_i)^j{}_k = -\mathbf{i}\epsilon_{ijk}$ satisfy the $\mathfrak{su}(2)$ algebra. These are the generators of the **3**-dimensional representation of $\mathsf{SU}(2)$.

   (b) What are the eigenvalues of $\hat{n} \cdot \vec{J}$, where $\hat{n}$ is a unit vector?

   (c) Find an explicit expression for $e^{\mathbf{i}\theta\hat{n}\cdot\vec{J}}$, where $\vec{J}$ is the vector of matrices above. Check that your answer is reasonable when $\hat{n} = \hat{z}$.

   Hint: Show that $(\hat{n} \cdot \vec{J})^3 = \hat{n} \cdot \vec{J}$ and use this to sum the exponential series. One way to see this is to use the previous part to conclude that $(J^3)^2$ is a projector.

4. **Conjugacy classes of rotations.**

   Consider a rotation matrix $O \in \mathsf{SO}(n)$. For purposes of this problem, take as given the fact that conjugation $O \to VOV^{-1}$ changes the axis of rotation, but not the angle.

   (a) What are the eigenvalues of a rotation matrix?

   (b) What are the eigenvectors of a rotation matrix?

   (c) Given a rotation matrix, how can you determine the angle of rotation?

   (d) Given a rotation matrix, how can you determine the plane of rotation?

   Hint: solve this problem in two dimensions first. Then for general n use conjugation to put the matrix in a canonical form.

5. **Counting solutions to algebraic equations in a finite group.** [Bonus problem]

(a) [This part you did already on the previous homework] Starting from the definition of the Frobenius-Schur indicator for a representation R_a, find a formula for $\sigma(h)$, the number of solutions of the equation $g^2 = h$ in a finite group G. Hint: multiply the definition by $\chi_a^\star(h)$, sum over a, and use character orthogonality.

Show that the number of square roots $\sigma(h)$ is a class function.

(b) Redo the previous problem to count solutions of $g^3 = h$ in G. Write the answer in terms of the object

$$\eta_a^{(3)} \equiv \frac{1}{|G|} \sum_{g \in G} \chi_a(g^3).$$

What's the number of cube roots of (12) in S_3?

(c) [Super-bonus problem] Show that $\eta_a^{(3)}$ defined above is nonzero only when there is a cubic invariant of the representation R_a, that is $R_a \otimes R_a \otimes R_a = \mathbf{1} \oplus \cdots$. Does it say something more specific (*e.g.* about how many cubic invariants there are)? Is $\eta_a^{(3)}$ always an integer?

(d) Count homomorphisms of the trefoil knot group $\langle g, h | h^2 = g^3 \rangle$ to S_3. [Hint: $\frac{1}{|G|} \sum_g D^a(g^2)$ is an intertwiner.]