University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 220 Symmetries Fall 2020
Assignment 5

Due 12:30pm Monday, November 9, 2020

Thanks for following the submission guidelines on hw 01. Please ask me by email if you have any trouble.

Relative to the first version of this problem set I posted, I moved a problem to the next problem set, and replaced it with the last one.

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1. **Using the algebra of classes to construct the character table for S_3.**

   (a) Write explicit expressions for the $\mathbf{C}_\alpha$ operators for the case $G = S_3$, and find the structure constants $c^\gamma_{\alpha\beta}$ of the algebra of classes. A nice way to represent this information is as a table

	$\mathbf{C}_1$	$\mathbf{C}_2$	$\mathbf{C}_3$
$\mathbf{C}_1$	$\mathbf{C}_1\mathbf{C}_1$	$\mathbf{C}_1\mathbf{C}_2$	$\mathbf{C}_1\mathbf{C}_3$
$\mathbf{C}_2$	$\mathbf{C}_2\mathbf{C}_1$	$\mathbf{C}_2\mathbf{C}_2$	$\mathbf{C}_2\mathbf{C}_3$
$\mathbf{C}_3$	$\mathbf{C}_3\mathbf{C}_1$	$\mathbf{C}_3\mathbf{C}_2$	$\mathbf{C}_3\mathbf{C}_3$

   (b) Construct the matrices $(\mathbf{C}_\alpha)^\gamma{}_\beta$, check that they commute, and find their eigenvalues, λ^a_α

   (c) Check that you reproduce the character table for S_3 using the fact that the eigenvalues should be $\lambda^a_\alpha = \chi^a_\alpha / \chi^a_e$.

2. **Fun with the Frobenius-Schur indicator.**

   (a) Compute the Frobenius-Schur indicator for the **2** of S_3 using the matrices you constructed on the last homework. If you get zero, can you find the basis where it is real?

   (b) Consider the 2-dimensional representation of Q_8. Using the character table, decide if it is complex, real or pseudoreal.

   Write down its representation matrices in terms of Pauli matrices and check your answer.

   (c) Let $\sigma_h \equiv$ the number of square roots g of the element $h = g^2$ in the group G. Then by writing the Frobenius-Schur indicator for an irrep as

$$\eta_a \equiv \frac{1}{|G|} \sum_{g \in G} \chi_a(g^2) = \frac{1}{|G|} \sum_{h=g^2 \in G} \sigma_h \chi_a(h) \qquad (1)$$

and using character orthogonality, show that

$$\sigma_h = \sum_{\text{irreps},a} \eta_a \chi_a(h). \tag{2}$$

3. **Fun with Frobenius reciprocity.**

   Take each irrep of S_3 and decompose it into irreps of $\mathbb{Z}_3 \subset S_3$ (for example using the characters).

   (b) For each of the irreps of $\mathbb{Z}_3$, construct the induced representation of S_3 (you did this already for the trivial representation) and decompose it into irreps of S_3 (for example using the characters).

   (c) Check Frobenius reciprocity.

   (d) [Bonus problem] Verify Frobenius reciprocity for $A_4 \subset S_4$.

4. **A_5 is simple.**

   (a) Show that an invariant subgroup $H \subset G$ is a union of conjugacy classes of G.

   (b) [Bonus problem] Find the number of elements of the conjugacy classes n_C of A_5. The answer is $n_C = 1, 15, 20, 12, 12$.

   (c) Show that A_5 is simple by checking for possible sums of n_Cs (necessarily including 1) which divide $5!/2$.

5. **Induced representations.**

   (a) Check that the definition of induced representation in the notes actually produces a representation of G in the sense that $D(g_1 g_2) |n, x\rangle = D(g_1)D(g_2) |n, x\rangle$ for all $g_{1,2} \in G$ and $|n, x\rangle \in W \times V_{G/H}$.

   (b) Show that the character of the induced representation $\mathrm{Ind}_H^G(W)$ is

   $$\chi_{\mathrm{Ind}_H^G(W)}(g) = \mathrm{Ind}_H^G[\chi_W](g)$$

   where the operation on the RHS is defined in the lecture notes.

   (c) Prove Frobenius reciprocity:

   $$\left\langle \psi, \mathrm{Res}_H^G[\phi] \right\rangle_H = \left\langle \mathrm{Ind}_H^G[\psi], \phi \right\rangle_G .$$