University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 220 Symmetries Fall 2020
Assignment 2

Due 12:30pm Monday, October 19, 2020

Thanks in advance for following the submission guidelines from hw 01. Please ask me by email if you have any trouble.

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1. **Brain-warmer.** What is the cycle structure of the permutation (235)(245)?

2. **Brain-warmer.**

   (a) What group is this: $G = \langle a, b | aba^{-1}b^{-1} = e\rangle$?

   (b) [Bonus problem] Find a space X so that $\pi_1(X) = G$ above.

3. **Quaternions.** [refugee from hw 1]

   Decompose the quaternion group Q_8 into conjugacy classes.

4. **Brain-warmer.** Check the relation $|G| = |Z(g)||C(g)|$ for $g = (12)$ in $G = S_n$. Here $Z(g)$ is the centralizer of g (the set of elements of G that commute with g) and $C(g)$ is the conjugacy class of g (by $|C(g)|$ I mean the number of elements in the conjugacy class).

5. **A presentation of A_4.** Prove that the group $\langle a, b | a^2 = e, b^3 = e, (ab)^3 = e\rangle$ is isomorphic to A_4, the group of even permutations of 4 objects,

   $A_4 = \{e, (12)(34), (14)(23), (13)(24), (123), (132), (243), (234), (341), (314), (421), (412)\}.$

6. **Free groups are weird.** [Bonus problem] Show that the free group on two elements $\langle a, b|\rangle$ contains subgroups isomorphic to the free group on any number of elements.

7. **Counting elements of conjugacy classes of S_k.** Here is a cool trick, related to Polya enumeration, for counting the number of elements in the conjugacy class of S_k associated to a given Young diagram (cycle structure), λ.

   (a) [bonus problem]. Fill in the missing details of the following argument. First, recall the object $z(\sigma) \equiv z_1^{c_1(\sigma)} z_2^{c_2(\sigma)} \cdots$, where $c_i(\sigma)$ is the number of cycles of length i in the permutation σ. This is a conjugation-invariant weight over which we can sum:

$$Z_G(z_1, z_2, \cdots) \equiv \sum_{\sigma \in S_k} z(\sigma).$$

This is (proportional to) the object we called the cycle index in our discussion of Polya enumeration (for the case with $G = S_k$).

Now consider the case where $|X| = k$ and $G = S_k$, the whole permutation group on the k objects, and we'll take n colors (*i.e.* an n-state Potts model on X). Weight a coloring with l_i objects of color i with a factor of $W = u_1^{l_1} u_2^{l_2} \cdots$. Polya's enumeration theorem says that the partition sum is then

$$\sum_{\text{orbits } O} W(O) = Z_{S_k}\left(z_1 = u_1 + u_2 + \cdots, z_2 = u_1^2 + u_2^2 + \cdots, \cdots\right) \tag{1}$$

$$= \sum_{l_1, l_2, \cdots l_n} (\#\text{ of orbits with } l_1 \text{ 1s, } l_2 \text{ 2s...})\, u_1^{l_1} u_2^{l_2} \cdots \tag{2}$$

What is this number of orbits? Because we are modding out by the whole permutation group, an orbit is entirely determined by specifying the number of each color. So this number is only ever 1 or 0.

To avoid the cases where it's zero, here's the final trick, familiar from statistical mechanics as the grand canonical ensemble: sum over k (!). Let

$$P(t, u_1, u_2, \cdots) \equiv \sum_{k=1}^{\infty} t^k \sum_{\text{orbits } O \text{ of } S_k} W(O).$$

On the one hand, this is

$$P(t, u_1, u_2, \cdots) = \sum_k t^k Z_{S_k}\left(z_1 = u_1 + u_2 + \cdots, z_2 = u_1^2 + u_2^2 + \cdots, \cdots\right).$$

On the other hand, this is

$$P(t, u_1, u_2, \cdots) = \sum_k t^k \sum_{l_1, l_2, \cdots l_n} (\#\text{ of orbits with } l_1 \text{ 1s, } l_2 \text{ 2s...})\, u_1^{l_1} u_2^{l_2} \cdots \tag{3}$$

$$= \sum_{l_1=0}^{\infty} (tu_1)^{l_1} \sum_{l_2=0}^{\infty} (tu_2)^{l_2} \cdots \tag{4}$$

$$= \exp\left(z_1 t + z_2 t^2/2 + z_3 t^3/3 + \cdots\right) \tag{5}$$

where $z_i \equiv u_1^i + u_2^i + \cdots$.

So for example, to compute the sizes of the conjugacy classes of S_7, let $T = z_1 t + z_2 t^2/2 + z_3 t^3/3 + \cdots$ (you can stop at some number bigger than 7), and just find the coefficient of t^7 in e^T. The result is a polynomial in the z_i where the sum of the subscripts of each term adds up to 7. Each term is then associated with a Young diagram λ and hence a conjugacy class.

(b) What should you get if you set $z_i = 1$ for all i and why?

(c) Find the size of each conjugacy class of S_4 and S_5. (I recommend Mathematica's Series and Coefficient commands and the method described in the previous part of the problem.) Check that your polynomial satisfies the check of the previous part.

(d) Actually, there is a better way to learn the sizes of conjugacy classes of S_n. The order of the centralizer of g, $|Z_g|$, depends only on its conjugacy class. The size of the conjugacy class is then $|C_g| = |G|/|Z_g|$ (by Lagrange's theorem). Show that the centralizer of an element g of S_n with c_j cycles of length j is

$$|Z_g| = \prod_j (c_j)! j^{c_j}. \tag{6}$$

[Hint: Think about what elements that commute with a permutation of a given cycle structure can do, then count them.] Write a formula for the number of elements of the conjugacy class C_g and compare with your results from the previous part.

8. **Counting non-isomorphic graphs.** A graph with k vertices can be regarded as a choice of $\{0, 1\}$ for each of the $\binom{k}{2} = \frac{k(k-1)}{2}$ pairs of vertices ('1' means no edge and '1' means yes edge). Two graphs are isomorphic if they are related by a relabelling of the vertices. How many non-isomorphic graphs on 4 vertices are there? (The result of the previous problem will be useful.)

   Construct the partition function which weights a graph by the number of edges, $Z(t) = \sum_{\text{graphs, }\Gamma} t^{\#\text{ of edges of }\Gamma}$ for $k = 4$.

   Bonus problem: answer the above questions for 5 vertices.

9. **Quotients of the spherical model.** [This is a bonus problem, since (a) it uses some perhaps-unfamiliar notions from statistical mechanics and (b) I just made it up from scratch, so it has not yet been road-tested.] I mentioned that one application of the counting theorems we proved might be to statistical mechanics. So far all the examples I've shown involve a finite number N of degrees of freedom. But statistical mechanics is most interesting in the thermodynamic limit, $N \to \infty$, where there can be phase transitions, symmetry-breaking and well-defined phases.

   (a) A simple, solvable model with a phase transition at finite temperature is the following. Place two-valued spins at each of N sites (think of them as arranged in a circle) and couple them by all-to-all ferromagnetic Ising interactions, so the hamiltonian is

$$H = -J \sum_{i,j=1}^{N} s_i s_j$$

where each sum is over all N spins. (The price of the solvability is that the model has no notion of locality. This will be an advantage for the purposes of this problem.) We will set $J = \frac{1}{N}$ so that there is a nice thermodynamic limit. The model has a $\mathbb{Z}_2$ symmetry which takes $s_i \to -s_i$.

The canonical partition function is

$$Z_0(\beta) = \sum_{\{s_i = \pm 1\}} e^{-\beta H(s)} = \sum_{\{s_i = \pm 1\}} e^{\beta \frac{1}{N} \sum_i s_i \sum_j s_j}.$$

Using the formula $\int_{-\infty}^{\infty} dx e^{-\frac{Nx^2}{4\beta} + xNM} = \sqrt{\frac{4\pi\beta}{N}} e^{-\beta N M^2}$ rewrite $Z_0(\beta)$ in a way so that the sum over spins factorizes. Then argue that at large N you can do the integral over x by saddle point, and do so. Find the two phases (distinguished by whether or not the $\mathbb{Z}_2$ symmetry is spontaneously broken) by a graphical analysis of the saddle point equation.

(b) In the model of the previous part, the phase transition arises by a competition between energy and entropy: it is energetically favorable for the spins to line up, but there are more configurations where the spins are not lined up, so (minimizing $F = E - TS$) disorder wins at high temperature.

Suppose we declare two spin configurations equivalent if they are related by a cyclic permutation of the spins (around the circle). Such transformations form the group $\mathbb{Z}_N$. (This process is called *gauging* this $\mathbb{Z}_N$ symmetry.) You might think this would favor the ordered phase, since the configurations where all spins point the same way is fixed by this transformation, while the shift relates several disordered configurations, and therefore reduces their number.

Using the not-Burnside lemma and its consequences, compute the partition function for the model with the same H as above, but where configurations are orbits under $\mathbb{Z}_N$. This is easiest when N is a prime number, so restrict to this case. Explain what property of $\mathbb{Z}_N$ for N prime makes this case easier.

(c) [A harder but really fun question, even more optional, since I had not entirely solved it yet at the time of posting] Does the conclusion of the previous part change if instead of gauging $\mathbb{Z}_N$, we gauge the whole S_N interchanging all the spins?