University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 239/139 Fall 2019
Assignment 7

Due 12:30pm Wednesday, November 20, 2019

1. **Two faces of strong subadditivity.** Show that for any state $\boldsymbol{\rho}_{ABC}$ on $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$,
$$S(A) + S(B) \leq S(AC) + S(BC)$$
and that this statement is equivalent to strong subadditivity in the form
$$S(A'C') + S(A'B') \geq S(A') + S(A'B'C') \qquad \forall A'B'C' \ .$$
Hint: use the Araki-Lieb purification trick.

2. **Consequences of SSA for mutual information.** Prove that SSA implies
$$I(A:B) + I(A:C) \leq 2S(A) \ .$$
Is the analogous inequality for Shannon entropies true?
Find an example of a state where $I(A:B) > S(A)$.

3. **Measurement is coarse-graining.**
Let $\boldsymbol{\rho}, \boldsymbol{\sigma}$ be two states on $\mathcal{H}$, and let $\{\mathcal{M}_x\}$ be a POVM. Define the classical probability distributions p_x, q_x from the outcomes of a measurement of $\{\mathcal{M}_x\}$ on the states $\boldsymbol{\rho}, \boldsymbol{\sigma}$ respectively (that is, $p_x = \mathrm{tr}\boldsymbol{\rho}\mathcal{M}_x$ etc). Show that
$$\hat{D}(\boldsymbol{\rho}||\boldsymbol{\sigma}) \geq D(p||q).$$

4. **Scramble.**
For this problem $\mathcal{H}_A$ has dimension d.
   (a) **Warmup.** The set of linear operators $\mathrm{End}(\mathcal{H}_A)$ is itself a Hilbert space with the Hilbert-Schmidt inner product $\langle \mathbf{A}, \mathbf{B} \rangle = \mathrm{tr}\mathbf{A}^\dagger \mathbf{B}$. Find an orthogonal basis $\{\mathbf{U}_a\}$ for this space (over $\mathbb{C}$) whose elements are themselves unitary operators, $\mathrm{tr}\mathbf{U}_a^\dagger \mathbf{U}_b = d\delta_{ab}$.
   [Hint: consider the algebra generated by the unitaries $\mathbf{X}, \mathbf{Z}$ on the qdit teleportation problem on the previous problem set.]
   Bonus: For the case of $|A| = 2^k$ find such a basis whose elements square to one.

(b) Consider a maximally entangled state $|\Phi\rangle \equiv \frac{1}{\sqrt{d}}\sum_i |ii\rangle \in \mathcal{H}_A \otimes \mathcal{H}_A$. Show that the d^2 maximally entangled states

$$|\Phi_a\rangle \equiv \mathbf{U}_a \otimes \mathbb{1}\,|\Phi\rangle$$

form an orthonormal basis of $\mathcal{H}_A \otimes \mathcal{H}_A$.

(c) Check your answers to the previous two parts for the case of qbits $d = 2$. Make a basis of product states from linear combinations of maximally entangled states.

(d) For an arbitrary operator $\mathbf{A} \in \mathrm{End}(A)$ find $\{p_a, \mathbf{U}_a\}$ with p_a probabilities and $\mathbf{U}_a$ unitary such that the associated channel *scrambles* $\mathbf{A}$ in the sense that

$$\sum_a p_a \mathbf{U}_a \mathbf{A} \mathbf{U}_a^\dagger = \frac{\mathrm{tr}\mathbf{A}}{d}\mathbb{1}.$$

(e) Use the previous result and the concavity of the entropy to show that the uniform state $\mathbf{u} = \mathbb{1}/d$ has the maximum von Neumann entropy on A.

(f) Bonus problem: for the case where $\mathbf{A}$ is Hermitian, find a set of only d unitaries which scramble $\mathbf{A}$.

5. **Random quantum expanders.** [somewhat open-ended and numerical]

Consider the family of quantum channels of the form

$$\boldsymbol{\rho} \mapsto \mathcal{E}_\chi(\boldsymbol{\rho}) = \sum_{i=1}^{\chi} p_i \mathbf{U}_i \boldsymbol{\rho} \mathbf{U}_i^\dagger$$

with $\{\mathbf{U}_i\}$ a collection of unitaries. Such a channel is called a *quantum expander*. Show that such a channel is unital.

Sample χ random unitaries from the Haar measure on $\mathsf{U}(d)$ *e.g.* in Mathematica[1]. (You can take $p_i = 1/\chi$ for definiteness if you wish.)

Sample a random initial density matrix[2].

Consider the rate at which repeated action of the channel $\mathcal{E}_\chi$, $\boldsymbol{\rho}_n = \mathcal{E}^n(\boldsymbol{\rho})$ mixes the initial state $\boldsymbol{\rho}$ as a function of χ (and d). We can use the von Neumann entropy as a measure of this mixing. Make some plots and some estimates.

[1] Haar measure means the measure which is invariant under the group action. I did this by choosing a $d \times d$ complex matrix X with entries chosen from the gaussian distribution (which is indeed invariant under $\mathsf{U}(d)$) and then taking $Y = X + X^\dagger$ to make it hermitian, and then using the matrix U which diagonalizes Y.

[2] I did this by choosing a complex matrix X with entries chosen from the gaussian distribution, and then taking $Y = X + X^\dagger$ to make it hermitian and then taking $Z = Y^2$ to make it positive and then taking $\boldsymbol{\rho} = Z/\mathrm{tr}Z$ to make it a density matrix. What distribution did I use?

If n is very large, how many terms do I actually need to include in the sum in

$$\mathcal{E}^n(\boldsymbol{\rho}) = \sum_{i_1..i_n} p_{i_n} \cdots p_{i_1} \mathbf{U}_{i_1} \cdots \mathbf{U}_{i_n} \boldsymbol{\rho} \mathbf{U}_{i_n}^\dagger \cdots \mathbf{U}_{i_1}^\dagger \quad ?$$

Consider the eigenstates (eigenoperators) of the (super)operator $\mathcal{E}_\chi$. Can you show that any state orthogonal (in the Hilbert-Schmidt norm) to $\mathbb{1}$ has a an eigenvalue less than 1?