University of California at San Diego – Department of Physics – Prof. John McGreevy

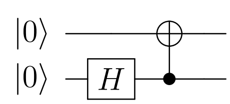
Physics 239/139 Fall 2019
Assignment 6

Due 12:30pm Wednesday, November 13, 2019

1. **Shannon entropy is concave.** Consider a collection of probability distributions π^α on a random variable X, so $\sum_x \pi^\alpha_x = 1, \pi^\alpha_x \geq 0, \forall x$. Then a convex combination of these $\pi_{\text{av}} \equiv \sum_\alpha p_\alpha \pi^\alpha$ is also a probability distribution on X. Show that the entropy of the average distribution is larger than the average of the entropies:
$$H(\pi_{\text{av}}) \geq \sum_\alpha p_\alpha H(\pi^\alpha).$$

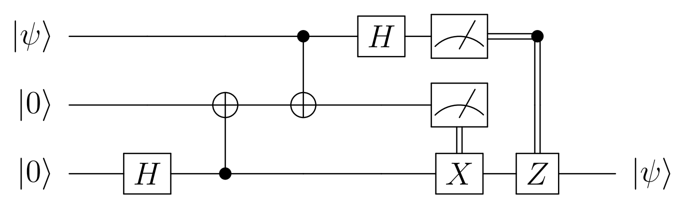
2. **Brainwarmer.** Check that the Holevo quantity $\chi(p_a, \rho_a) = S(\sum_a p_a \rho_a) - \sum_a p_a S(\rho_a)$ can be written as a relative entropy
$$\chi(p_a, \rho_a) = D(\rho_{AB} || \rho_A \otimes \rho_B)$$
with $\rho_{AB} \equiv \sum_a p_a \rho_a \otimes |a\rangle\langle a|$ where $B = \text{span}\{|a\rangle\}$ and the $|a\rangle$ are orthonormal.

3. **Making a Bell pair from a product state.** Find the output of the following quantum circuit (time goes to the right here and in the following):



Here $H = \frac{1}{\sqrt{2}}(X + Z)$ is a Hadamard gate, and the two-qbit gate is the CX gate as in lecture.

4. **Quantum Teleportation.** Convince yourself that it is possible to transmit an unknown state of a qbit by sending two classical bits to someone with whom you share a Bell pair, using the following circuit:



Time goes from left to right here; you should recognize the first two operations from the previous problem. Imagine that the register on the bottom line is separated in space from the top two after this point. The measurement boxes indicate measurements of Z; the double lines indicate that the outcomes of these measurements $s = 1, 0$ (this is the sending of the two classical bits) determine whether or not (respectively) to act with the indicated gate.

5. **Quantum Dense Coding.** Find a circuit which does the reverse of the previous: by sending an unknown qbit to someone with whom you share a Bell pair, transmit two classical bits. (Hint: basically just reverse everything in the previous problem.)

6. **Teleportation for qdits.** [optional]

   Show that it is possible to teleport a state $|\xi\rangle_A \in \mathcal{H}_A, |A| \equiv d$ from A to B using the maximally-entangled state

$$|\Phi\rangle_{AB} \equiv \frac{1}{\sqrt{d}} \sum_{n=1}^{d} |nn\rangle_{AB} \,.$$

   Hint: Consider the clock and shift operators

$$\mathbf{Z} \equiv \sum_{n=1}^{d} |n\rangle \langle n| \, \omega^n, \ \omega \equiv e^{\frac{2\pi \mathbf{i}}{d}}, \quad \mathbf{X} \equiv \sum_{n=1}^{d} |n+1\rangle \langle n|$$

   where the argument of the ket is to be understood mod d. Show that these generalize some of the properties of the Pauli $\mathbf{X}$ and $\mathbf{Z}$ in that they are unitary and that they satisfy the (discrete) Heisenberg algebra

$$\mathbf{XZ} = a\mathbf{ZX}$$

   for some c-number a which you should determine.

7. **Conditional entropy in terms of relative entropy.**

   (a) Show that the conditional entropy can be written as

$$S(A|C) = -D(\rho_{AC} || \mathbb{1}_A \otimes \rho_C). \tag{1}$$

   (b) Does the relation (1) imply that the conditional entropy is always negative? Find a proof or a counterexample.

8. **It's a trap.** Is the mutual information convex?

$$I_{\sum_a p_a \rho_a}(A:B) \overset{?}{\leq} \sum_a p_a I_{\rho_a}(A:B)$$

   It's a relative entropy, and the relative entropy is jointly convex in its arguments, right? Find a proof or a counterexample.