University of California at San Diego – Department of Physics – Prof. John McGreevy

# Physics 239/139 Fall 2019
# Assignment 4

**Due 12:30pm Monday, October 28, 2019**

1. **Error correcting code brain-warmer.**

   (a) For the [7,4] Hamming code discussed in lecture, check that $Ht = 0$ where $t$ is any codeword, and $H$ is the given parity check matrix.

   (b) If you ($B$) are communicating with someone ($A$) through channel with $H(A|B) = 1/7$ using this code and you receive the string

   $$r = (0, 1, 0, 0, 0, 0, 1)^t,$$

   what is the most likely intended message string?

2. **Chain rule for mutual information.** [optional]

   Show from the definitions that the mutual information satisfies the following chain rule:

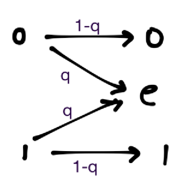
   $$I(X : YZ) = I(X : Y) + I(X : Z|Y) = I(X : Z) + I(X : Y|Z).$$

   More generally,

   $$I(X_1 \cdots X_n : Y) = \sum_{i=1}^{n} I(X_i Y|X_{i-1} \cdots X_1). \quad (1)$$

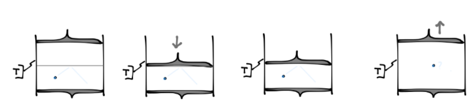
3. **Binary erasure channel.**

   Find the channel capacity of this channel:




4. **Mechanical engineering problem.** [optional]

In lecture I claimed that the expansion of an ideal gas against a piston, as in the figure at right, could be used to lift a weight.



Design a plausible system of strings and pulleys to make this happen.

5. **Test of Landauer's Principle.** [optional]

   Consider logical bits which are stored in the magnetization (up or down) of little magnets. Show that copying a known bit (say 0) onto an *unkown* bit by the method described in lecture costs energy at least $k_B T \ln 2$.

6. **Control-X brainwarmer.**

   Show that the operator control-X can be written variously as

$$\mathsf{CX}_{BA} = |0\rangle\langle 0|_B \otimes \mathbb{1}_A + |1\rangle\langle 1|_B \otimes \mathbf{X}_A = \mathbf{X}_A^{\frac{1}{2}(1-\mathbf{Z}_B)} = e^{\frac{\mathbf{i}\pi}{4}(1-\mathbf{Z}_B)(1-\mathbf{X}_A)}.$$

7. **Density matrix exercises.**

   (a) Show that the most general density matrix for a single qbit lies in the Bloch ball, *i.e.* is of the form

$$\boldsymbol{\rho}_v = \frac{1}{2}\left(\mathbb{1} + \vec{v}\cdot\vec{\boldsymbol{\sigma}}\right), \quad \sum_i v_i^2 \leq 1.$$

   Find the determinant, trace, and von Neumann entropy of $\boldsymbol{\rho}_v$.

   (b) [from Barnett] A single qbit state has $\langle \mathbf{X}\rangle = s$. Find the most general forms for the corresponding density operator with the minimum and maximum von Neumann entropy. (Hint: the Bloch ball is your friend.)

   (c) Show that the *purity* of a density matrix $\pi[\boldsymbol{\rho}] \equiv \mathrm{tr}\boldsymbol{\rho}^2$ satisfies $\pi[\boldsymbol{\rho}] \leq 1$ with saturation only if $\boldsymbol{\rho}$ is pure.

   (d) [from Barnett] Show that the quantum relative entropy satisfies the following

$$D(\boldsymbol{\rho}_A \otimes \boldsymbol{\rho}_B || \boldsymbol{\sigma}_A \otimes \boldsymbol{\sigma}_B) = D(\boldsymbol{\rho}_A || \boldsymbol{\sigma}_A) + D(\boldsymbol{\rho}_B || \boldsymbol{\sigma}_B). \tag{2}$$

$$\sum_i p_i D\left(\boldsymbol{\sigma}_i || \boldsymbol{\rho}\right) = \sum_i p_i D(\boldsymbol{\sigma}_i || \boldsymbol{\sigma}_{\mathrm{av}}) + D(\boldsymbol{\sigma}_{\mathrm{av}} || \boldsymbol{\rho}) \tag{3}$$

$$D\left(\boldsymbol{\sigma}_{\mathrm{av}} || \boldsymbol{\rho}\right) \leq \sum_i p_i D(\boldsymbol{\sigma}_i || \boldsymbol{\rho}) \tag{4}$$

   for any probability distribution $\{p_i\}$ and density matrices $\boldsymbol{\rho}, \boldsymbol{\sigma}_i$, and where $\boldsymbol{\sigma}_{\mathrm{av}} \equiv \sum_i p_i \boldsymbol{\sigma}_i$.

8. **Thermal density matrix.** Suppose given a Hamiltonian $H$. In lecture we showed that the thermal density matrix $\boldsymbol{\rho}_T \equiv \frac{e^{-\frac{H}{k_B T}}}{Z}$ has the maximum von Neumann entropy for any state with the same expected energy. Show that if instead we are given a fixed temperature $T$, the thermal density matrix minimizes the free energy functional

$$F_T[\boldsymbol{\rho}] \equiv \mathrm{tr}\boldsymbol{\rho} H - T S_{vN}[\boldsymbol{\rho}].$$