University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 239/139 Fall 2019
Assignment 3

Due 12:30pm Monday, October 21, 2019

1. **Mutual information bounds correlations.** Consider again the distribution on two binary variables from last homework: $p_{yx} = \begin{pmatrix} 0 & a \\ b & b \end{pmatrix}_{yx}$, where $y = 1, -1$ is the row index and $x = 1, -1$ is the column index (so yx are like the indices on a matrix). Normalization implies $\sum_{xy} p_{xy} = a + 2b = 1$, so we have a one-parameter family of distributions, labelled by b.

   (a) I've changed the labels on the variables from $\uparrow, \downarrow$ to $1, -1$ so that we can consider correlation functions, such as the connected two-point function

   $$C \equiv \langle xy \rangle_c \equiv \langle xy \rangle - \langle x \rangle \langle y \rangle$$

   where $\langle A \rangle \equiv \sum_{xy} p_{yx} A$. Compute C as a function of b.

   (b) Compute the mutual information between X and Y

   $$I(X:Y) = \sum_{xy} p_{yx} \log \frac{p_{yx}}{p_y p_x}.$$

   (c) Check that

   $$I(X:Y) \geq \frac{1}{2} C^2$$

   for every value of b (for example, plot both functions).

   (d) [Bonus] The inequality I quoted in lecture, and which we will prove in the more general quantum case later, is

   $$I(X:Y) \geq \frac{1}{2} \frac{\langle \mathcal{O}_X \mathcal{O}_Y \rangle_c^2}{\| \mathcal{O}_X \|^2 \| \mathcal{O}_Y \|}$$

   where the norms are defined (in the classical case) by

   $$\| \mathcal{O}_X \|^2 \equiv \sup_{p | \sum_x p_x = 1} \{ \sum_x \mathcal{O}_x^\star \mathcal{O}_x p_x \}.$$

   Show that in the above example, the 'operators' x, y are normalized, in the sense that $\| x \| = \| y \| = 1$.

2. **Strong subadditivity, the classical case.** [From Barnett] Prove *strong subaddivity* of the Shannon entropy: for any distribution on three random variables,

$$H(ABC) + H(B) \leq H(AB) + H(BC).$$

(The corresponding statement about the von Neumann entropy is not quite so easy to show.)

Hint: $q(a, b, c) \equiv \frac{p(a,b)p(b,c)}{p(b)}$ is a perfectly cromulent probability distribution on ABC.

What is the name for the situation when equality holds? Write the condition for equality in terms of the conditional mutual information $I(A : C|B)$.

3. **Symbol coding problem.** You are a mad scientist, but a sloppy one. You have 127 identical-looking jars of liquid, and you have forgotten which one is the poison one. You have at your disposal 7 rats on whom your poor moral compass will allow you to test the liquids. However (the rats have a strong social network and excellent spies) you only get one shot: the rats must drink all at once (or they will catch on to what is happening and revolt). You may mix the liquids in separate containers. Any rat that drinks any amount of poison will turn bright orange. Design a protocol to uniquely identify the poison jar.

4. **Another coding problem.** [optional, but how can you resist?] The problem is to establish a code by which you can transmit to your friend a number from $1 \cdots N = 64$. The tool you will be given is a chessboard (8×8) where an adversary has randomly placed identical markers on some of the squares. You are only allowed to add or remove a single marker. Your friend will see only the result of your actions, not the initial configuration. You may speak to your friend beforehand.

5. **Huffman code.** Make the Huffman code for the probability distribution $p(x) = \{.5, .2, .15, .1, .05\}$.

Compare the average word length to the Shannon entropy.

Bonus: what property of the distribution determines the deviation from optimality?

6. **Huffman code decryption problem.** [Optional, but fun.]

```
0010 11000100000101011 010101010010110 00011001000011 00110100111
101010010001100011 111110000101000 0001011001010010100011010111O111
11101000100 000101000000100011101011011O1101.  1111100011
```

```
00011001010101101010111001011001111 1
00010101001000100000101010001101011
0101101000011000100001100111110010010000111000 1010101001001
1100101101010110101101110000001 1 0110011010110100000110010001101 0100
0101100000011 1101011000000011 1110100 1000010010111011
0101101000011101011011010000010010000 11.  0000110110000100,
00011001010101101010111001011001111 1 1110110001001001111000
1010010010110 11101 01010000100001000011
1000110100100101001100111110101010 1
00011001010101101010111001011001111 1
1001000000110001001100100011010100 1000 010010101 1101011111110111001
10101100101100010010111011001100110101100100111000.
```

Hint: I used the letter frequencies from *The Origin of Species*.

You might want to use `Mathematica` to do this problem!

7. **Analogy with strong-disorder RG.** [open ended, more optional question]

   Test or decide the following consequence suggested by the analogy between Huffman coding and strong-disorder RG: The optimality of the Huffman code is better when the distribution is broader. A special case is the claim that the Huffman code is worst when all the probabilities are the same. Note that the outcome of the Huffman algorithm in this case depends on the number of elements of the alphabet.

   Measure the optimality by $\langle \ell \rangle - H[p]$ (or maybe $\frac{\langle \ell \rangle - H[p]}{H[p]}$?).

8. **Binary symmetric channel.**

   For the binary symmetric channel ABE defined in lecture, with $a, b, e \in \{0, 1\}$, and

   $$p(a) = (p, 1-p)_a, \; p(e) = (q, 1-q)_e, \quad \text{and} \quad b = (a+e)_2,$$

   find all the quantities $p(a, b), p(b), p(b|a), p(a|b)$ and $H(B), H(B|A), I(B : A), I(B : A|E)$. Find the channel capacity.